

Laws and Likelihoods for Ornstein Uhlenbeck-Gamma and other BNS OU Stochastic Volatility models with extensions.

LANCELOT F. JAMES¹

The Hong Kong University of Science and Technology

In recent years there have been many proposals as flexible alternatives to Gaussian based continuous time stochastic volatility models. A great deal of these models employ positive Lévy processes. Among these are the attractive non-Gaussian positive Ornstein-Uhlenbeck (OU) processes proposed by Barndorff-Nielsen and Shephard (BNS) in a series of papers. One current problem of these approaches is the unavailability of a tractable likelihood based statistical analysis for the returns of financial assets. This paper, while focusing on the BNS models, develops general theory for the implementation of statistical inference for a host of models. Specifically we show how to reduce the infinite-dimensional process based models to finite, albeit high, dimensional ones. Inference can then be based on Monte Carlo methods. As highlights, specific to BNS we show that an OU process driven by an infinite activity Gamma process, that is an OU- Γ , exhibits unique features which allows one to exactly sample from relevant joint distributions. We show that this is a consequence of the OU structure and the unique calculus of Gamma and Dirichlet processes. Owing to another connection between Gamma/Dirichlet processes and the theory of Generalized Gamma Convolutions (GGC) we identify a large class of models, we call (FGGC), where one can perfectly sample marginal distributions relevant to option pricing and Monte Carlo likelihood analysis. This involves a curious result, we establish as Theorem 6.1. We also discuss analytic techniques and candidate densities for Monte-Carlo procedures which can be applied to more general classes of models.

Contents

1	Introduction	2
1.1	Proposal and outline	4
2	Preliminaries	5
2.1	Some more preliminary OU results	5
2.2	BNS Likelihood model	6
2.3	Some points about GGC and Gamma processes	7
2.3.1	Connection to Cifarelli and Regazzini distribution theory	8
2.3.2	Perfect simulation of $M_{\theta H}$	8
3	Laws and Likelihoods for the OU-Γ model	9
3.1	Perfect simulation of $M_{\theta a}$	13
3.2	BNS OU- Γ likelihood inference	13
3.3	The likelihood via a connection to Variance Gamma processes	14
3.4	Bayesian estimation and related comments	16

¹Supported in part by grants HIA05/06.BM03 and DAG04/05.BM56 of the HKSAR.
AMS 2000 subject classifications. Primary 62G05; secondary 62F15.
Corresponding authors address. The Hong Kong University of Science and Technology, Department of Information and Systems Management, Clear Water Bay, Kowloon, Hong Kong. lance@ust.hk
Keywords and phrases. Bessel Functions, Dilogarithm function, Dirichlet Process, Ornstein-Uhlenbeck Process, Perfect Sampling, Stochastic Volatility, Weber-Sonine Formula.

3.5	OU- Γ processes with possibly random scale parameter	17
3.6	Likelihoods for Superpositioned OU- Γ	17
3.7	Randomly sampled times	18
3.7.1	Time changed Integrated OU- Γ processes	18
3.8	The special nature of the OU- Γ process as an SV model	19
4	General Likelihoods	20
4.1	Fourier-Cosine integral representation of the likelihood	20
5	General OU likelihoods	21
6	Some distribution theory for OU-FGGC models	22
6.1	Results for perfect sampling relevant OU-FGGC components	22
6.2	OU-FGGC Monte Carlo Densities	24
6.3	OU-FGGC option pricing densities	25
7	Some practical issues for general OU likelihood estimation	25
7.1	Calculating Lévy exponents	25
7.2	Monte Carlo method	26
7.2.1	Sampling from Q_1	27
7.2.2	Sampling from Q_2	27
8	General approach	28
9	Examples	29
9.1	OU-Stable	29
9.2	IG-OU	30
9.3	OU-LogNormal	30
9.4	OU-FGGC where H is the Arcsine distribution	30

1 Introduction

Barndorff-Nielsen and Shephard (2001a, b)(BNS) introduce a class of continuous time stochastic volatility (SV) models that allows for more flexibility over Gaussian based models such as the Black-Scholes model[see Black and Scholes (1973) and Merton (1973)]. Their proposed SV model is based on the following differential equation,

$$(1) \quad dx^*(t) = (\mu + \beta v(t))dt + v^{1/2}(t)dw(t)$$

where $x^*(t)$ denotes the log-price level, $w(t)$ is Brownian motion, and independent of $w(t)$, $v(t)$ is a stationary Non-Gaussian Ornstein-Uhlenbeck (OU) process which models the *instantaneous volatility*. This latter point is equivalent to the fact that for $\lambda > 0$,

$$v(t) = e^{-\lambda t}v(0) + e^{-\lambda t} \int_0^t e^{\lambda y} Z(dy)$$

and arises as the solution of the following differential equation,

$$dv(t) = -\lambda v(t) + dZ(\lambda t).$$

In the above framework Z is a positive homogeneous process, otherwise known as a *subordinator*, on $[0, \infty)$ and $v(0)$ is an arbitrary positive random variable independent of Z . That is $Z(t) := \int_0^t Z(dy)$ is a stationary process, with $Z(0) = 0$, and its distribution specified by its Laplace transform for each $\omega > 0$,

$$(2) \quad \mathbb{E}[e^{-\omega Z(t)}] = e^{-t\psi(\omega)}$$

where $\psi(\omega) = \int_0^\infty (1 - e^{-s\omega})\rho(ds)$, is often called the *Lévy exponent* of an infinite divisible random variable equivalent in distribution to $Z(1)$, and ρ is its corresponding Lévy density. Either of these characterizes the distribution of the process Z . Importantly, it is obvious from (2), that one does not need explicit knowledge of ρ to calculate ψ . Note further that if we wish $v(t)$ to be *stationary* it is necessary to choose $v(0) \stackrel{d}{=} \int_{-\infty}^0 e^s Z^*(ds)$, where Z^* is independent of Z but otherwise has the same law.

The model described above is an extension of the Black-Scholes or Samuelson model which arises by replacing v with a fixed variance, say σ^2 . The additional innovation in BNS is that modeling volatility as a random process, $v(t)$, rather than a random variable, not only allows for heavy-tailed models, but additionally induces serial dependence. This serial dependence is used to account for a clustering affect referred to as *volatility persistence*. The work of Carr, Geman, Madan, and Yor (2003) discuss this point further. See also Duan (1995) and Engle (1982) for different approaches to this type of phenomenon. The model of BNS has gained a great deal of interest with some related works including Carr, Geman, Madan, and Yor (2003), Barndorff-Nielsen and Shephard (2003), Eberlein (2001), Nicolato and Venardos (2001), Benth, Karlsen, and Reikvam (2003). See also the discussion section in Barndorff-Nielsen and Shephard (2001a). See Carr and Wu (2004) and Duffie, Pan and Singleton (2000) for many other models.

Note that the log price at time t is $x^*(t) = \mu t + \beta\tau(t) + \tau^{1/2}(t)w(t)$ where

$$\tau(t) = \int_0^t v(s)ds = \lambda^{-1}[(1 - e^{-\lambda t})v(0) + \int_0^t (1 - e^{-\lambda(t-y)})Z(d\lambda y)]$$

is referred to as a *integrated* OU process and models the integrated variance. Quantities of interest are often based on the aggregate returns, for $s < t$, $x^*(t) - x^*(s)$ which involves

$$(3) \quad \tau(t) - \tau(s) = \lambda^{-1}[(1 - e^{-\lambda(t-s)})v(s) + \int_s^t (1 - e^{-\lambda(t-y)})Z(d\lambda y)]$$

where again importantly, $v(s) = (e^{-\lambda s}v(0) + \int_0^s e^{-\lambda(s-y)}Z(d\lambda y))$.

Barndorff Nielsen and Shephard (2001a, Section 5.4.1 and 6.2) show that laws related to the random functions

$$(4) \quad (Z(\lambda t), e^{-\lambda t} \int_0^t e^{\lambda y} Z(d\lambda y))$$

play a key role both in option pricing and likelihood estimation. Specifically option pricing requires some type of description of the distribution of

$$(5) \quad \int_s^t (1 - e^{-\lambda(t-y)})Z(d\lambda y) \stackrel{d}{=} \int_0^\Delta (1 - e^{-y})Z(dy),$$

for $\Delta = (t - s) > 0$. Although the density of (5) is not often known in a nice closed form one can apply inversion techniques via its characteristic function or Laplace transform which is described in BNS (2001a, 2003).

However, as seen in BNS (2001a, 5.4) it is a rather challenging problem to find tractable approaches to statistical analysis of likelihood models based on n aggregate returns, $X_i = x^*(i\Delta) - x^*((i-1)\Delta)$ over periods of time $[(i-1)\Delta, i\Delta]$ for $i = 1, \dots, n$ and $\Delta > 0$. [This framework can be extended to intervals of varying lengths say Δ_i]. These models are based on the unobserved *actual variances* $\tau_i = \tau(i\Delta) - \tau((i-1)\Delta)$ for $i = 1, \dots, n$. It is easy to see that one may write

$$(6) \quad X_i = \mu\Delta + \beta\tau_i + \tau_i^{1/2}\epsilon_i$$

where ϵ_i for $i = 1, \dots, n$ are independent standard Normal random variables. Hence it follows that conditional on (τ_1, \dots, τ_n) the X_i are independent Normal random variables with unknown mean $\mu\Delta + \beta\tau_i$ and variance τ_i . The major obstacle to tractable analysis of such models is that in general the joint distribution of (τ_1, \dots, τ_n) is rather complex. BNS (2001a, 5.4) show that statistical inference can be done if one were able to sample, *efficiently*, n iid copies of the pair

$$(7) \quad (Z(\lambda\Delta), e^{-\lambda\Delta} \int_0^\Delta e^{\lambda y} Z(dy)).$$

The problem is that it is not obvious how to deal with the joint distributional behavior of the above pair (7). This is in contrast to the option pricing problem which essentially involves the distribution of a single random variable. A generic, theoretically all purpose, approach is to use an infinite series representation. Several MCMC procedures, based on variations of this idea, have been proposed to handle subclasses of these models requiring simulation of points from random processes. See for instance, Roberts, Papaspiliopoulos and Dellaportas (2004) and Griffin and Steel (2005), who use compound Poisson process specifications for Z , and the discussion section in Barndorff-Nielsen and Shephard (2001a). For approaches to other types of models see for instance Eraker, Johannes, and Polson (2003).

While these methods have their attractive points they do not provide exact solutions for cases where Z is an *infinite activity* process, such as a Gamma process or more generally a Generalized Inverse Gamma (GIG) process. Moreover these methods are computationally non-trivial and further work needs to be done to assess their accuracy for different processes. Another important point is that they cannot be used if one does not have specific knowledge of the Lévy density associated with Z . This excludes for instance the case where Z is based on a Pareto or LogNormal distribution.

1.1 Proposal and outline

This paper focuses on several subtopics related to the issues above. In particular we discuss methods that avoid working directly with infinite dimensional components. First, perhaps most remarkably, we will show that if one chooses Z to be a Gamma process then one can sample exactly random variables based on the pair in (7) and (4). In addition, we will be able to derive the explicit density of certain quantities which is also relevant to option pricing. Curiously we will show that the explicit densities depend on the *dilogarithm* function

$$Li_2(x) := - \int_0^x \frac{\log(1-u)}{u} du := \sum_{k=1}^{\infty} \frac{x^k}{k^2}$$

The dilogarithm function is a well-studied special function that arises often in a variety of contexts. See for instance Maximon (2003) and Flajolet and Sedgewick (2006). This leads to an explicit description of the relevant τ_i in terms of sums of independent random variables which allows one to perform likelihood estimation based on sampling $2n$ iid random random variables as well as the independent random variable $v(0)$. We then easily extend this framework to possibly random observation times. An important point is that these results allow one to also use τ in other likelihood models not discussed in BNS (2001a, 5.4). These facts have not been pointed out in the literature.

They are derived from the unique properties of the Gamma/Dirichlet process calculus wherein we are able to exploit a, not immediately obvious, connection to Dirichlet Process mean functionals. In as much, the seminal work of Cifarelli and Regazzini (1990) and the perfect simulation methods discussed in Guglielmi, Holmes and Walker (2002) play a key role. The corresponding OU process $v(t)$ is known as OU- Γ process. This should not be confused with the often discussed Γ -OU process where the BDLP is a compound Poisson process and $v(t)$ has Gamma distributed marginals. Also, our results suggest that one could simply use the OU- Γ as a building block for more intricate models.

Secondly all the properties that we exploit for the Gamma case do not extend to other OU models. However we show that the ability to perfectly sample the marginal distributions of quantities relevant to option pricing and likelihood estimation extends to a large class of models where Z is a Generalized Gamma Convolution (GGC). We call these models finite GGC or (FGGC). A highlight of this paper related to this class of models is Theorem 6.1.

Although we shall focus primarily on the BNS OU models, we note that there are many others which can be found for instance in Carr and Wu (2004)[see also Carr, Geman, Madan and Yor (2003)]. As such we shall employ an analytical technique which leads to an expression of the relevant likelihoods in terms of an n -dimensional Fourier-Cosine integral. This technique is loosely based on the ideas in James (2005b). Multidimensional Fourier-Cosine integral appear often in various fields including physics. We then focus on ingredients necessary to carry out Monte Carlo procedures which are known to be well suited to approximating high-dimensional integrals. More details may be obtained from the provided table of contents.

REMARK 1. Throughout, when appropriate, we will be describing the law of a generic positive random variable W by its corresponding Lévy exponent defined as

$$-\log E[e^{-\omega W}]$$

We will use the notation Δ as an arbitrary positive distance between two points. We shall specify its value when necessary, i.e. $\Delta = t - s$, $\Delta = t$ and so on. We will also often use the notation $a = \lambda\Delta$.

2 Preliminaries

This paper utilizes results from several linked but not often jointly studied areas. We anticipate that the average reader will be familiar with some but not all of the topics. As such we provide some details that we shall exploit. The majority of the discussion in sections 2.1-2.2 may be found in BNS (2001a,b, 2003). Section 2.3 is again a blend of ideas from several fields.

2.1 Some more preliminary OU results

We will describe the distribution of pertinent quantities via their Lévy exponents, and discuss the basic structure of the likelihood. First note that for any positive g on $[0, \infty)$, we may define a random variable $Z(g) := \int_0^\infty g(x)Z(dx)$. Moreover it is fairly well-known that the Lévy exponent of $Z(g)$ is given by $\int_0^\infty \psi(\omega g(x))dx$. It is clear that all the OU related processes that we encounter are representable as some $Z(g)$ where $g(x)$ is readily identified. Using this fact or consulting directly BNS(2001a,b, 2003) one has that the Lévy exponent of the quantity in (5) is

$$(8) \quad \int_{e^{-\lambda\Delta}}^1 \psi(\omega(1-u))u^{-1}du$$

for $\Delta = (t - s) > 0$. The Lévy exponent of $e^{-\lambda t} \int_0^t e^{\lambda y} Z(d\lambda y) \stackrel{d}{=} e^{-\lambda\Delta} \int_0^\Delta e^{\lambda y} Z(d\lambda y)$ for $\Delta = t$ is,

$$\int_{e^{-\lambda\Delta}}^1 \psi(\omega u)u^{-1}du$$

If we wish to choose $v(t)$ stationary then the Lévy exponent of $v(0)$ must be

$$(9) \quad \int_0^\infty \psi(\omega e^{-s}) ds = \int_0^1 \psi(\omega u) u^{-1} du.$$

2.2 BNS Likelihood model

The model of Barndorff-Nielsen and Shephard (2001a, section 5.4) translates into a likelihood based model as follows. Let X_i for $i = 1, \dots, n$ denote a sequence of aggregate returns of the log price of a stock observed over intervals of length $\Delta > 0$, described in (6). Suppose additionally the Z depends on unknown parameters ϱ . The likelihood of the model depends on unknown parameters $\vartheta = (\mu, \beta, \lambda, \varrho)$ and as stated before the $X_i | \vartheta, \tau$ are iid Normal random variables. Ideally one is interested in estimating ϑ based on the likelihood

$$(10) \quad \mathcal{L}(\mathbf{X} | \vartheta) = \int_{\mathbb{R}_+^n} \left[\prod_{i=1}^n \phi(X_i | \mu\Delta + \beta\tau_i, \tau_i) \right] f(\tau_1, \dots, \tau_n | \varrho, \lambda) d\tau_1, \dots, d\tau_n$$

where, setting $A_i = (X_i - \mu\Delta)$, and $\bar{A} = n^{-1} \sum_{i=1}^n A_i$,

$$\phi(X_i | \mu\Delta + \beta\tau_i, \tau_i) = e^{A_i\beta} \frac{1}{\sqrt{2\pi}} \tau_i^{-1/2} e^{-A_i^2/(2\tau_i)} e^{-\tau_i\beta^2/2}$$

denotes a Normal density. The quantity $f(\tau_1, \dots, \tau_n | \varrho, \lambda)$ denotes the joint density of the integrated volatility based on the intervals $[(i-1)\Delta, i\Delta]$ for $i = 1, \dots, n$. Barndorff-Nielsen and Shephard (2001a) note that the likelihood is intractable and hence makes exact inference difficult. The apparent intractability is attributed to the complex nature of $f(\tau_1, \dots, \tau_n | \varrho, \lambda)$ which is derived from a random measure. Specifically, the BNS models complexities arises from the following structure of the τ_i . From (3) one has for the BNS model

$$(11) \quad \lambda\tau_i = (1 - e^{-\lambda\Delta})v((i-1)\Delta) + \int_{(i-1)\Delta}^{i\Delta} (1 - e^{-\lambda(i\Delta-y)})Z(dy)$$

where importantly for $r_j = e^{\lambda j\Delta}$, and

$$O_j = \int_{(j-1)\Delta}^{j\Delta} e^{-\lambda(j\Delta-y)}Z(dy),$$

$v((i-1)\Delta) = e^{-\lambda(i-1)\Delta}[v(0) + \sum_{j=1}^{i-1} r_j O_j]$. It is not difficult to see that the O_j are iid for $j = 1, \dots, n$ but are correlated with corresponding terms

$$\int_{(j-1)\Delta}^{j\Delta} (1 - e^{-\lambda(j\Delta-y)})Z(dy) = Z_j - O_j$$

where $Z_j := [Z(\lambda j\Delta) - Z(\lambda(j-1)\Delta)] \stackrel{d}{=} Z(\lambda\Delta)$. Furthermore O_l appears in each τ_i for $i \geq l$. Hence the suggestion by BNS to try to sample the iid pairs in (7).

Indeed the joint distribution of the (τ_1, \dots, τ_n) is in general complex. However one can easily obtain its joint Laplace transform. It is with this fact that we argue that the primary stumbling block which currently prevents one from integrating out the infinite-dimensional components in the likelihood, is inherent from the Normal distribution of $X_i | \vartheta, \tau$. Quite simply the Normal assumption yields exponential terms of the form

$$e^{-A_i^2/(2\tau_i)} \text{ rather than } e^{-\tau_i A_i^2}.$$

We will show in the forthcoming sections how to apply a Bessel integral representation, which does not depend on the distribution of (τ_1, \dots, τ_n) to obtain expressions for likelihood based on quite general candidates for τ . First however we will describe the very remarkable and unique properties of the OU- Γ model in section 3 which does not require this approach.

2.3 Some points about GGC and Gamma processes

We will be making extensive use of the basic elements of the theory of Generalized Gamma Convolutions (GGC) which can be found in Bondesson (1979, 1992) and Thorin (1977). GGC are a sub-class of infinitely divisible random variables. A nice point is that they all have the important *self-decomposability* property. This has an interesting consequence since it is well known that $v(t)$ is a stationary OU process if and only if $v(0) \stackrel{d}{=} v(t)$ is self-decomposable. See for instance Wolfe (1992), Jurek and Vervaat (1983), Sato (1999), Jeanblanc, Pitman and Yor (2002) or BNS (2001a, Theorem 1) for a more precise statement. That is, there is a large subclass of OU models which all have GGC laws. Some important examples of GGC random variables and corresponding processes are GIG laws, Stable laws of index $0 < \alpha < 1$, and of course Gamma random variables.

Important, from our point of view, is that a random variable is a GGC if and only if its Lévy exponent is expressible as

$$(12) \quad \int_0^\infty d_\theta(\omega x) \nu(dx)$$

for some arbitrary sigma-finite measure satisfying appropriate conditions so that (12) is finite and where

$$(13) \quad d_\theta(\omega) = \theta \log(1 + \omega) = \int_0^\infty (1 - e^{-\omega s}) \theta s^{-1} e^{-s} ds.$$

corresponding to the Lévy exponent of a Gamma random variable with shape parameter θ . That is to say d_θ is a special case of ψ and moreover the Lévy density of a corresponding Gamma process is given by

$$\rho_\theta(ds) = \theta s^{-1} e^{-s} ds \text{ for } s > 0$$

It then follows that the Lévy density of a GGC is given by

$$(14) \quad \theta s^{-1} \int_0^\infty e^{-s/r} \nu(dr).$$

As a consequence, if we denote a Gamma process on a Polish space \mathcal{X} with sigma finite shape measure $\theta \nu^*$ as $G_{\theta \nu^*}$, then (12) is significant as it coincides with the Lévy exponent of an arbitrary Gamma process mean functional say $G_{\theta \nu^*}(g) = \int_{\mathcal{X}} g(x) G_{\theta \nu^*}(dx)$ where by a change of variable, $R = g(X)$, one can write equivalently in distribution as $\int_0^\infty r G_{\theta \nu}(dr)$.

Now we point to a key fact that has not been exploited much in the literature. First throughout this paper let T_θ denote a Gamma random variable with shape θ and scale 1. Denote the density of a Gamma random variable with shape θ and scale $b > 0$ as

$$\mathcal{G}_\theta(y|b) = \frac{b^{-\theta}}{\Gamma(\theta)} y^{\theta-1} e^{-y/b} \text{ for } y > 0.$$

When $b = 1$, we simply write $\mathcal{G}_\theta(y)$. Let (J_i) denote the jump points of a Gamma process and let (Z_i) denote the points of a Poisson random measure whose laws are determined by ν , which are independent of (J_i) . It is well known that one can write $G_{\theta \nu}(dx) = \sum_{i=1}^\infty J_i \delta_{Z_i}(dx)$. Furthermore, it follows that if Y is a GGC random variable then one can always write

$$Y \stackrel{d}{=} G_{\theta \nu^*}(g) \stackrel{d}{=} T_\theta M_{\theta \nu}$$

where $M_{\theta \nu} = \sum_{i=1}^\infty (J_i/T_\theta) Z_i$, is a random variable independent of T_θ . The independence property is due to the known fact that the sequence (J_i/T_θ) of probabilities is independent of T_θ which may be written as $T_\theta = \sum_{j=1}^\infty J_j$. This property uniquely characterizes a Gamma process and has nothing to do with whether or not ν is finite or more generally sigma finite. The sequence (J_i/T_θ) is known to have the Poisson-Dirichlet law.

Hence when $\nu := H$ is a finite measure, which we will take without loss of generality to be a probability measure, a Dirichlet Process with shape θH , having total mass $\theta H(\mathcal{X}) = \theta$, is defined by the representations

$$P_{\theta H}(dx) := \frac{G_{\theta H}(dx)}{G_{\theta H}(\mathcal{X})} = \sum_{j=1}^{\infty} \frac{J_j}{T_{\theta}} \delta_{Z_j}(dx)$$

where importantly $T_{\theta} = G_{\theta H}(\mathcal{X})$ is independent of $P_{\theta H}$. Setting $\mathcal{X} = [0, \infty)$ one has that

$$M_{\theta H} = \int_0^{\infty} x P_{\theta H}(dx)$$

is a Dirichlet Process mean functional which again is independent of T_{θ} . This independent property naturally comes from the finite dimensional Beta-Gamma calculus based on the classic result of Lukacs (1955), which we shall also use. That is, if T_{θ_i} for $i = 1, \dots, n$ are independent Gamma random variables with shape θ_i then the sum, $\sum_{i=1}^n T_{\theta_i} = T_{\theta^*}$, where $\theta^* = \sum_{i=1}^n \theta_i$ and moreover is independent of the vector of probabilities $(T_{\theta_i}/T_{\theta^*})$ which has the Dirichlet distribution with density

$$D(p_1, \dots, p_n) \propto \prod_{i=1}^n p_i^{\theta_i - 1}$$

where the $\sum_{i=1}^n p_i = 1$. We will denote the fact that a random vector has Dirichlet law of this type by writing $\text{DIRICHLET}_n(\theta_1, \dots, \theta_n)$. Similarly denote a two parameter beta law as $\text{BETA}(\theta_1, \theta_2)$.

2.3.1 Connection to Cifarelli and Regazzini distribution theory

Because of these observations we are able to exploit the works of Cifarelli and Regazzini (1990) and those of subsequent authors to obtain expressions for the marginal densities of relevant components of large class of models which we call OU-FGGC. The FGGC are models with Lévy density defined by (14) with $\nu := H$. This is relevant to both option pricing and likelihood estimation. We should add that many of these properties will extend to more general moving average models where Z is an FGGC BDLP.

The study of properties of Dirichlet process mean functional has been a major area of interest in Bayesian Nonparametrics. This line of work was initiated by the paper of Cifarelli and Regazzini (1990). One of their important contributions was to obtain explicit expressions for densities of mean functionals $M_{\theta H}$. Let $f_{M_{\theta H}}$ denote the density of $M_{\theta H}$. Set $H(x) = \int_0^x H(du)$. Then from Cifarelli and Regazzini (1990) or Cifarelli and Melilli (2000) one has for $\theta = 1$

$$(15) \quad f_{M_H}(x) = \frac{1}{\pi} \sin(\pi H(x)) e^{-\int_0^{\infty} \log(|t-x|) H(dt)}$$

and when $\theta > 1$,

$$(16) \quad f_{M_{\theta H}}(x) = \frac{\theta - 1}{\pi} \int_0^x (x - u)^{\theta-2} \frac{1}{\pi} \sin(\pi \theta H(u)) e^{-\theta \int_0^{\infty} \log(|t-u|) H(dt)} du$$

One can also obtain an expression for the cdf of $M_{\theta H}$ that holds for all $\theta > 0$, we do not list that here.

2.3.2 Perfect simulation of $M_{\theta H}$

It is evident that given the form of the density in (15) one can in principle use some sort of rejection sampling procedure to obtain realizations of M_H . With a bit more care one can devise an efficient method to sample $M_{\theta H}$ for $\theta > 1$ using the density in (16). Importantly, as pointed out by Hjort

and Ongaro (2005), when $\theta = m$, where $m = 2, 3, \dots$, is an integer one can use (15) to sample M_{mH} based on the following fact,

$$M_{mH} \stackrel{d}{=} \sum_{i=1}^m P_i M_{1,i}$$

where $(M_{1,i})$ are iid with common distribution equivalent to M_H given by (15). Moreover (P_1, \dots, P_n) is independent of $(M_{1,i})$ and is n -dimensional $\text{DIRICHLET}_n(1, \dots, 1)$. This can be seen as a simple consequence of the infinite divisibility of $T_m M_{mH}$, where, as a consequence, $T_m M_{mH} \stackrel{d}{=} \sum_{i=1}^m T_{1,i} M_{1,i}$, and applying the Beta Gamma calculus. That is further writing $P_i = T_{1,i}/T_m$, where $T_m = \sum_{j=1}^m T_{1,j}$ is independent of (P_i) . What is important is that these methods do not rely on the more computationally burdensome, and otherwise approximate, series methods. There is however yet another approach which will allow one to easily *perfectly sample* $M_{\theta H}$ for all $\theta > 0$.

Recently, in the case where $M_{\theta H}$ is almost sure bounded, Guglielmi, Holmes and Walker (2002) devise a very simple and efficient method to obtain perfect samples from the distribution of $M_{\theta H}$ that works for all $\theta > 0$. We recount the basic elements of that algorithm. First note that $0 \leq a \leq M_{\theta H} \leq b$ if and only if the support of H is $[a, b]$. As explained in Guglielmi, Holmes and Walker (2002), following the procedure of Propp and Wilson (1996), one can design an upper and lower dominating chain starting at some time $-N$ in the past up to time 0. The upper chain, say $uM_{\theta H}$, is started at $uM_{\theta H, -N} = b$, and the lower chain, $lM_{\theta H}$, is started at $lM_{\theta H, -N} = a$. One runs the Markov chains for each n based on the equations,

$$(17) \quad uM_{\theta H, n+1} = B_{n, \theta} X_n + (1 - B_{n, \theta}) uM_{\theta H, n}$$

and

$$(18) \quad lM_{\theta H, n+1} = B_{n, \theta} X_n + (1 - B_{n, \theta}) lM_{\theta H, n}$$

where the chains are coupled using the same random independent pairs $(B_{n, \theta}, X_n)$ where for each n , $B_{n, \theta}$ has a $\text{Beta}(1, \theta)$ distribution and X_n has distribution H . The chains are said to coalesce when $D = |uM_{\theta H, n} - lM_{\theta H, n}| < \epsilon$ for some small ϵ . Notice importantly that this method only requires knowledge of the distribution H .

REMARK 2. Vershik, Tsilevich and Yor (2004) and James (2005a) are two examples of applications that directly exploit the independence property exhibited at the level of the Gamma/Dirichlet process. See also Diaconis and Kemperman (1996) and Diaconis and Freedman (1999) for more interesting facts.

REMARK 3. More discussion on the merits of self-decomposability as it relates to financial applications can be found in Carr, Geman Madan and Yor (2005).

3 Laws and Likelihoods for the OU- Γ model

For $\theta > 0$, define a OU- Γ process by setting $Z = G_\theta$, where G_θ denote a homogeneous Gamma process on $[0, \infty)$, i.e. $\nu(dx) = dx$ for $x \in [0, \infty)$ with law specified by its Lévy exponent $d_\theta(\omega)$ given in (13). Letting $v_\theta(t)$ denote the stationary OU- Γ it follows that its Lévy exponent is

$$(19) \quad \int_0^1 d_\theta(\omega u) u^{-1} du = \theta \int_0^\infty (1 - e^{-\omega y}) y^{-1} E_1(y) dy = -\theta Li_2(-\omega)$$

where $E_1(y) = \int_y^\infty e^{-u} u^{-1} du = \int_1^\infty e^{-uy} u^{-1} du$ is *Euler's exponential integral*. That is to say the Lévy density of $v_\theta(0)$ is $\rho_{v_\theta}(dy) = \theta y^{-1} E_1(y) dy$.

REMARK 4. In addition to obtaining the form of the Lévy density, BNS (2003, p.283) note that the Lévy exponent of a OU- Γ can be expressed as,

$$\theta \sum_{j=1}^{\infty} (-1)^j \frac{\omega^j}{j^2} \text{ for } 0 \leq \omega < 1$$

but they don't equate this with the dilogarithm function.

The previous discussion indicates that one can implement both option pricing and likelihood analysis if one can sample the special case of (7) given by

$$(G_{\theta}(\lambda\Delta), e^{-\lambda\Delta} \int_0^{\Delta} e^{\lambda y} G_{\theta}(d\lambda y)).$$

The Lévy exponent of the second term is given by

$$\int_{e^{-a}}^1 d_{\theta}(\omega u) u^{-1} du = \int_{e^{-a}}^1 d_{\theta a}(\omega u) F_a(du) = -\theta [Li_2(-\omega) - Li_2(-\omega e^{-a})]$$

where

$$(20) \quad F_a(y) = \int_{e^{-a}}^y \frac{1}{au} du = \frac{\log(y) + \alpha}{a}$$

is a cdf for $e^{-a} \leq y \leq 1$. However due to the fact $G_{\theta}(\lambda\Delta) \stackrel{d}{=} G_{\theta a F_a}([e^{-a}, 1]) \stackrel{d}{=} T_{\theta a}$ this is equivalent to sampling the pair

$$(T_{\theta a}, \int_{e^{-a}}^1 x P_{\theta a F_a}(dx))$$

where for $e^{-a} \leq y \leq 1$

$$P_{\theta a F_a}(dy) = \frac{G_{\theta a F_a}(dy)}{T_{\theta a}}$$

is a Dirichlet process random probability measure with shape parameter $\theta a F_a$.

REMARK 5. We shall use the notation $M_{\theta a}$ rather than the perhaps more accurate $M_{\theta a F_a}$ where it is understood that F_a is defined in (20)

We discuss some of the implications of these facts in the next two propositions.

Proposition 3.1 *For each fixed $\Delta > 0$, and $a = \lambda\Delta$ set $Y_{\theta a} := e^{-a} \int_0^{\Delta} e^{\lambda y} G_{\theta}(d\lambda y)$, where G_{θ} is a homogeneous Gamma process. It follows that $G_{\theta}(a) = \int_0^t G_{\theta}(d\lambda y) \stackrel{d}{=} T_{\theta a}$. Additionally, the following distributional properties hold.*

- (i) *Let $M_{\theta a} := \int_0^1 x P_{\theta a F_a}(dx)$ denote a Dirichlet process mean functional based on the shape parameter $\theta a F_a$. Then for each fixed Δ , one has the coordinate-wise equivalence in joint distribution,*

$$(G_{\theta}(a), Y_{\theta a}) \stackrel{d}{=} (T_{\theta a}, T_{\theta a} M_{\theta a})$$

where $M_{\theta a}$ is independent of $T_{\theta a}$. Furthermore $e^{-a} \leq M_{\theta a} \leq 1$ almost surely.

- (ii) $\int_0^t (1 - e^{-\lambda(\Delta-y)}) G_{\theta}(d\lambda y) = G_{\theta}(a) - Y_{\theta a} \stackrel{d}{=} T_{\theta a} [1 - M_{\theta a}]$

- (iii) $(G_{\theta}(a) - Y_{\theta a}, Y_{\theta a}) \stackrel{d}{=} (T_{\theta a} [1 - M_{\theta a}], T_{\theta a} M_{\theta a})$

□

PROOF. The result is already established by our construction and appealing to the unique independence property of the Gamma/Dirichlet process. However since the joint equivalence in statement (i) is the key factor separating the OU- Γ from other OU processes, hence quite delicate, we will check it via joint Laplace transforms. Evaluating the joint Laplace transform of the $(G_\theta(a), Y_{\theta a})$ at points (ω_1, ω_2) , it is easily seen that the joint Lévy exponent is

$$\int_{e^{-a}}^1 d_{\theta a}(\omega_1 + \omega_2 u) F_a(du).$$

Now being careful to use only the independence property of $T_{\theta a}$ and $M_{\theta a}$ and the fact that $M_{\theta a}$ is a Dirichlet process mean functional we proceed as follows. Write $\omega_1 T_{\theta a} + \omega_2 T_{\theta a} M_{\theta a} = T_{\theta a}[\omega_1 + \omega_2 M_{\theta a}] := W$. Furthermore note the $\omega_1 + \omega_2 M_{\theta a} = \int_0^1 (\omega_1 + \omega_2 x) P_{\theta a F_a}(dx) = P_{\theta a F_a}(g)$, for $g(x) = \omega_1 + \omega_2 x$. Now by independence of $T_{\theta a}$ and $M_{\theta a}$ the joint Laplace transform, taking expectation with respect to the Gamma law first is,

$$\mathbb{E}[e^{-W}] = \mathbb{E}[(1 + \omega_1 + \omega_2 M_{\theta a})^{-\theta a}] = \mathbb{E}[(1 + P_{\theta a F_a}(g))^{-\theta a}]$$

Now appealing to the well-known identity of Cifarelli and Regazzini (1990) it follows that

$$\mathbb{E}[(1 + P_{\theta a F_a}(g))^{-\theta a}] = \mathbb{E}[e^{-G_{\theta a F_a}(g)}]$$

which is the desired result. The above argument indeed establishes the proof but the very special nature of the result perhaps will not be fully clear until one reads section 3.8.□

The next result describes the distribution of $v_\theta(0)$ in the stationary case.

Proposition 3.2 *Let $v_\theta(0)$ have distribution described by the Lévy exponent (19). Let $G_{\theta\nu}$ denote a (non-finite) Gamma process on $[0, 1]$ with $\nu(du) = u^{-1}du$ where $\int_0^1 \nu(du) = \infty$. Then $v_\theta(0)$ is a generalized Gamma convolution (GGC) such that*

$$v_\theta(0) \stackrel{d}{=} \int_0^1 x G_{\theta\nu}(dx) \stackrel{d}{=} T_\theta \tilde{M}_\theta,$$

where $\tilde{M}_\theta = M_{\theta\nu}$ is independent of T_θ but is not a Dirichlet process mean functional. Furthermore, for each fixed θ , the distribution of \tilde{M}_θ is characterized by its generalized Cauchy-Stieltjes transform,

$$\mathbb{E}[e^{-\omega v_\theta(0)}] = \mathbb{E}[(1 + \omega \tilde{M}_\theta)^{-\theta}] = e^{\theta Li_2(-\omega)}$$

□

REMARK 6. It is quite possible to obtain an explicit form of the density of $v_\theta(0)$ by using standard inversion results for characteristic functions and noting the relationship of the complex valued dilogarithm function to the *Inverse Tangent Integral*,

$$Ti_2(y) = \int_0^y \frac{\arctan(u)}{u} du,$$

which is the imaginary part of the complex valued dilogarithm function, and *Clausen's Function*. For more details see Maximon (2003).

Recapping, Proposition 3.1 shows that the distribution of $(G_\theta(a), Y_{\theta a})$ is determined by the distribution of the independent random variables $(T_{\theta a}, M_{\theta a})$. Among OU processes discussed here, the independence property is unique to OU- Γ processes. Additionally, as we shall see this pair may be sampled exactly due to the fact that $M_{\theta a F_a}$ is a Dirichlet process mean functional. On the other hand Proposition 3.2 shows that although $v_\theta(0) \stackrel{d}{=} T_\theta \tilde{M}_\theta$ is a GGC, the results for the Dirichlet process do not apply to \tilde{M}_θ and we otherwise do not have a tractable expression for the explicit density of $v_\theta(0)$. However, we do believe that a careful use of the relationships mentioned in Remark 6 will lead to an explicit form. The next proposition, using the work of Cifarelli and Regazzini (1990), provides more details for the distribution of $M_{\theta a}$ and shows also that one can use the Dirichlet process results to obtain a good approximate for the distribution of $v_\theta(0)$.

Proposition 3.3 *For each $0 < a = \lambda\Delta < \infty$ and $\theta > 0$, let $Y_{\theta a} \stackrel{d}{=} e^{-\lambda\Delta} \int_0^\Delta e^{\lambda y} G_\theta(d\lambda y)$ denote an infinitely divisible random variable with Lévy exponent,*

$$\int_{e^{-a}}^1 d_\theta(\omega u) u^{-1} du = \int_{e^{-a}}^1 d_{\theta a}(\omega u) F_a(du) = -\theta[Li_2(-\omega) - Li_2(-\omega e^{-a})]$$

where $F_a(du) = a^{-1}u^{-1}du$ is the density of a random variable taking its values in the interval $[e^{-a}, 1]$. Then the following results hold

- (i) $Y_{\theta a} \stackrel{d}{=} T_{\theta a} M_{\theta a}$, where $M_{\theta a} = \int_{e^{-a}}^1 x P_{\theta a F_a}(dx)$ is a Dirichlet process mean functional.
- (ii) The Lévy density of $Y_{\theta a}$ is $\rho_{\theta a}(dy) = \theta y^{-1}[E_1(y) - E_1(ye^a)]dy$. Hence the cumulants of $Y_{\theta a}$ are for each integer j ,

$$\theta \int_0^\infty y^{j-1}[E_1(y) - E_1(ye^a)]dy = \theta \frac{\Gamma(j)}{j} (1 - e^{-a^j})$$

- (iii) When $\theta a = 1$, the density of M_1 is given by

$$(21) \quad \frac{1}{\pi} \sin\left(\left[\frac{-\pi \log(x)}{a}\right]\right) x^{\frac{1}{a}[1-\log(x)]-1} e^{\frac{\pi^2}{3a}} e^{\frac{-1}{a}[Li_2(x)+Li_2(\frac{e^{-a}}{x})]},$$

for $e^{-a} \leq x \leq 1$.

- (iv) When $\theta a = 1$, the density of $V_a := -\log(M_1)/a$ is given by

$$(22) \quad \frac{1}{\pi} \sin(\pi v) e^{-[v+v^2]} e^{\frac{\pi^2}{3a}} e^{\frac{-1}{a}[Li_2(e^{-av})+Li_2(e^{-a(1-v)})]},$$

for $0 \leq v \leq 1$.

- (v) When $\theta a > 1$, the density of $M_{\theta a}/a$ is given by

$$(23) \quad \frac{\theta a - 1}{\pi} \int_{-\log x/a}^1 (x - e^{-va})^{\theta a - 2} \sin(\pi \theta a v) e^{-\theta a[v+v^2]} e^{\frac{\theta \pi^2}{3}} e^{-\theta[Li_2(e^{-av})+Li_2(e^{-a(1-v)})]},$$

for $0 \leq v \leq 1$.

- (vi) If $\theta a = m$, where $m = 2, 3, \dots$ is an integer, then $M_m \stackrel{d}{=} \sum_{i=1}^m W_i M_{1,i}$, where $(M_{1,i})$ are iid with density (21) and independent of $(M_{1,i})$, $(W_i = T_{1,i} / \sum_{j=1}^m T_{1,j})$, where $T_{1,i} \stackrel{d}{=} T_1$ are iid, is a $\text{DIRICHLET}_m(1, \dots, 1)$ m -dimensional vector.

- (vii) $Y_{\theta a}$ converges in distribution to $v_\theta(0)$ as $e^{-a} \rightarrow 0$.

□

PROOF. Most of the results are immediate from our previous discussions. The forms of the density arises from application of Cifarelli and Regazzini (1990) which amounts to explicitly calculating $\int_{e^{-a}}^1 \log(|t-x|) F_a(dt)$ expressed in terms of the dilogarithm function. □

The last result in this section gives a completely tractable description of the conditional distribution of the log asset price at time t given information up to time s . This type of result is pertinent to option pricing as discussed in BNS(2001a, 6.2) and Nicolato and Vernados (2003).

Proposition 3.4 *Let $x_\theta^*(t)$ be defined as in (1) with $Z = G_\theta$. Additionally for $0 \leq s < t$, set $\Delta = (t-s)$ and define $h(\Delta, s) = (1 - e^{-\lambda\Delta})v_\theta(s)$ and $\mu_s^* = \mu\Delta + x_\theta^*(s) + \beta h(\Delta, s)$. Then the conditional density of $x_\theta^*(t)|x_\theta^*(s), v_\theta(s)$ is given by*

$$\int_0^\infty \phi(x|\mu_s^* + \beta y, h(\Delta, s) + y) q_{\theta a}(y) dy$$

where $q_{\theta a}(y) = \int_{e^{-a}}^1 \mathcal{G}_{\theta a}(y|(1-v)) f_{M_{\theta a}}(v) dv$. When $\theta a = 1$,

$$q_1(y) = \int_0^1 \mathcal{G}_1(y|(1-e^{-v})) f_{V_a}(v) dv$$

where f_{V_a} is the density of V_a given in (22). □

3.1 Perfect simulation of $M_{\theta a}$

Due to the fact that the dilogarithm function, $Li_2(x)$, is a well-understood special function, which is available in many computational packages, it is evident that the densities in (21) and (22) can be exactly sampled using a rejection procedure. Again based on the discussion in section 2.3.2 Statement (vi) of Proposition 3.3 shows that one can use this fact to easily obtain samples of M_m , and hence Y_m , for any integer m . With a bit more care one can devise an efficient method to sample $M_{\theta a}$ for $\theta a > 1$ using the density in (23). One can also use the perfect sampling method described in 2.3.2 for all θa , based on $uM_{\theta a, -N} = 1$ and $lM_{\theta a, -N} = e^{-a}$, $B_{n, \theta a}$ is BETA $(1, \theta a)$ and X_n has distribution F_a

3.2 BNS OU- Γ likelihood inference

The results in the previous section now give the ingredients to perform likelihood based statistical inference via simple exact sampling. Here we describe a bit more about the distribution of τ_i in the OU- Γ case and then extend the discussion to randomly sampled times.

Proposition 3.5 *Define for $\Delta > 0$, $a = \lambda\Delta$ and $i = 1, \dots, n$ $\tau_{\theta, i} := \tau_\theta(i\Delta) - \tau_\theta((i-1)\Delta)$, by setting $Z = G_\theta$ in (3). Furthermore, let $r_i = e^{\lambda i\Delta}$ for $i = 1, \dots, n$, with $r_0 = 1$. Then it follows that, for $i = 1, \dots, n$,*

$$(24) \quad \lambda\tau_{\theta, i} = (1 - e^{-\lambda\Delta})v_\theta((i-1)\Delta) + T_i[1 - M_i]$$

with,

$$v_\theta((i-1)\Delta) = e^{-\lambda(i-1)\Delta} [v_\theta(0) + \sum_{j=1}^{i-1} r_j T_j M_j]$$

where (T_i, M_i) are iid pairs independent of $v_\theta(0)$. Additionally, for each fixed i , T_i and M_i are independent with distributions specified by $T_i \stackrel{d}{=} T_{\theta a}$ and $M_i \stackrel{d}{=} M_{\theta a}$. This implies that likelihood inference for the model (10) may be obtained from the joint distribution of $(X_i, T_i, M_i, v_\theta(0))$ given by

$$(25) \quad \left[\prod_{i=1}^n \phi(X_i | \mu\Delta + \beta\tau_{\theta,i}, \tau_{\theta,i}) \mathcal{G}_{\theta a}(t_i) f_M(v_i) \right] f_{v_\theta(0)}(w)$$

where $\tau_{\theta,i}$ is expressed as in (24), with $T_i = t_i, M_i = v_i$, and $v_\theta(0) = w$. \square

A Bayesian procedure, which involves placing a prior on $\vartheta = (\mu, \beta, \lambda, \varrho)$, is quite natural and otherwise proceeds by standard arguments, in this setting. That is letting $\pi(\vartheta)$ denote a prior joint density it follows that a posterior distribution of $\vartheta | \mathbf{X}$ is determined by a posterior distribution of $\vartheta, (T_i, M_i), v_\theta(0) | \mathbf{X}$ which is proportional to

$$\pi(\vartheta) \left[\prod_{i=1}^n \phi(X_i | \mu\Delta + \beta\tau_{\theta,i}, \tau_{\theta,i}) \mathcal{G}_{\theta a}(t_i) f_M(v_i) \right] f_{v_\theta(0)}(w)$$

REMARK 7. The likelihood in (10) for the OU- Γ case obviously is obtained by integrating out the pertinent independent quantities in (25). Due to the Gamma distributions, the answer could be expressed in terms of integrals with respect to modified Bessel functions. Or otherwise a subclass of Generalized Inverse Gaussian (GIG) random variables.

REMARK 8. Note that in practice we can approximate a draw from the distribution of $v_\theta(0)$ by using instead $Y_{\theta\delta}$ for $e^{-\delta}$ small. Otherwise, if strict stationarity $v_\theta(t)$ is not a concern, one can certainly use any positive distribution for $v_\theta(0)$.

3.3 The likelihood via a connection to Variance Gamma processes

Recall in the stationary case that according to Proposition 3.2. $v_\theta(0) \stackrel{d}{=} T_\theta \tilde{M}_\theta$, where \tilde{M}_θ is not a Dirichlet process mean functional. However this point allows one to write τ_θ and $(\tau_{\theta,1}, \dots, \tau_{\theta,n})$ in terms of a product of a Gamma random variable and another independent random variable. Specifically, for $a = \lambda\Delta$, one may write

$$\tau_{\theta,i} = T_{\theta(1+na)} S_i$$

where for $i = 1, \dots, n$

$$\lambda S_i = (1 - e^{-a}) e^{-a(i-1)} \left[\frac{T_\theta}{T_{\theta(1+na)}} \tilde{M}_\theta + \sum_{j=1}^{i-1} \frac{T_j}{T_{\theta(1+na)}} r_j M_j \right] + \frac{T_i}{T_{\theta(1+na)}} [1 - M_i].$$

The vector $\mathbf{S} = (S_1, \dots, S_n)$ is independent of $T_{\theta(1+na)}$ which can be written as $T_\theta + \sum_{i=1}^n T_i$. We may also write

$$\lambda S_i = (1 - e^{-a}) e^{-a(i-1)} \left[P_{n+1} \tilde{M}_\theta + \sum_{j=1}^{i-1} P_j r_j M_j \right] + P_i [1 - M_i].$$

where $P_{n+1} = 1 - \sum_{j=1}^n P_j$, and (P_1, \dots, P_{n+1}) is $\text{DIRICHLET}_{n+1}(\theta a, \dots, \theta a, \theta)$ independent of all other random variables. Recall now that a $GIG(\nu, \delta, \gamma)$ random variable has density given by

$$g(x | \nu, \delta, \gamma) = \frac{(\gamma/\delta)^\nu}{2K_\nu(\delta\gamma)} x^{\nu-1} e^{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)} \text{ for } x > 0$$

where K_ν is a modified Bessel function. Recall also that $K_\nu(x) = K_{-\nu}(x)$.

Additionally we will exploit the following nice feature of $K_\nu(x)$. Suppose that for a $m = 0, 1, 2, \dots$, the $|\nu| = m + 1/2$, where $|\nu|$ denotes absolute value, then we can use the fact that

$$(26) \quad K_{m+1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \sum_{k=0}^m \frac{(m+k)!}{k!(m-k)!2^k} x^{-k}$$

See for instance Pitman (1999, eq. (40)) for a probabilistic interpretation of (26).

This facts leads to the following description of the likelihood.

Theorem 3.1 *The observations according to (6) can be represented as $X_i = \mu\Delta + \beta T_{\theta(1+na)} S_i + [T_{\theta(1+na)} S_i]^{1/2} \epsilon_i$, in the OU- Γ case. Setting $\gamma^2 = [2 + \beta^2 \sum_{j=1}^n S_j]$ and $\delta^2 = \sum_{j=1}^n A_j^2 / (2S_j)$, $\kappa = \theta(1+na)$ and $\nu = \kappa - n/2$, and $a = \lambda\Delta$. The following results hold.*

(i) *The likelihood in (10) can be written as,*

$$\mathcal{L}(\mathbf{X}|\vartheta) = e^{n\bar{A}\beta} \mathbb{E}_\vartheta \left[\frac{2K_\nu(\delta\gamma)}{(\gamma/\delta)^\nu \Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_i^{-1/2} \right]$$

(ii) *If θ and a are chosen such $|\nu| = m + 1/2$, for $m = 0, 1, 2, \dots$, then*

$$\mathcal{L}(\mathbf{X}|\vartheta) = e^{n\bar{A}\beta} \sum_{k=0}^m \frac{(m+k)!}{k!(m-k)!2^k} \mathbb{E}_\vartheta \left[e^{-\delta\gamma} \frac{2(\delta\gamma)^{-k} \gamma^{-1}}{(\gamma/\delta)^m \Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_i^{-1/2} \right] \sqrt{\frac{\pi}{2}}$$

As a special case $|\nu| = m + 1/2$ for all n , if $\theta a = 1/2$ and $\theta = m + 1/2$.

(iii) *If additionally $m = 0$, that is $\theta = 1/2$ and $a = 1$, then*

$$\mathcal{L}(\mathbf{X}|\vartheta) = e^{n\bar{A}\beta} \mathbb{E}_\vartheta \left[e^{-\delta\gamma} \frac{2\gamma^{-1}}{\Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_i^{-1/2} \right] \sqrt{\frac{\pi}{2}}$$

In all cases the distribution of (S_1, \dots, S_n) is completely determined by the $2n + 2$ independent random variables with joint density $[\prod_{i=1}^n \mathcal{G}_{\theta a}(t_i) f_{M_{\theta a}}(v_i)] \mathcal{G}_\theta(t) f_{\tilde{M}_\theta}(w) \square$

The next result in effect serves to make clear Theorem 3.1 but also highlights the possibility, from a practical point of view, for more data augmentation procedures

Proposition 3.6 *Consider the setup and notation in Theorem 3.1. Additionally define $\beta_*^2 = \beta^2 \sum_{j=1}^n S_j$. Then it is clear that*

$$\frac{2K_\nu(\delta\gamma)}{(\gamma/\delta)^\nu \Gamma(\kappa)} = \int_0^\infty y^{-\frac{n}{2}} e^{-\frac{1}{2}(\delta^2 y^{-1} + \beta_*^2 y)} \mathcal{G}_\kappa(y) dy,$$

which leads to another expression of the likelihood $\mathcal{L}(\mathbf{X}|\vartheta)$. Thus statistical inference may be based on simulation from the joint density

$$\mathcal{G}_\kappa(y) \left[\prod_{i=1}^n \mathcal{G}_{\theta a}(t_i) f_{M_{\theta a}}(v_i) \right] \mathcal{G}_\theta(t) f_{\tilde{M}_\theta}(w).$$

Based on this fact one has that if a random variable V has the density \mathcal{G}_κ relative to the representation of $\mathcal{L}(\mathbf{X}|\vartheta)$ then the posterior distribution of $V|\mathbf{S}, \mathbf{X}, \vartheta$ is $GIG(\nu, \delta, \gamma)$ with parameters specified by Theorem 3.1. and Proposition 3.5

REMARK 9. One notes that the expressions in statements (ii) and (iii) of Theorem 3.1 are quite manageable. Here one is perhaps taking the view that θ and $\lambda\Delta$ are chosen to ease computations. However note that in statement (ii) that m , whose parameter space is $\{0, 1, \dots\}$ becomes a viable and flexible parameter of interest from a modelling point of view. The expression in statement (i) is also quite amenable to Monte-Carlo estimation approaches.

REMARK 10. By Variance Gamma processes we are loosely referring to the work of Madan, Carr and Chang (1998), see also Carr, Geman, Madan, and Yor (2003). It is evident that all OU-GGC models exhibit similar properties. That is if the BDLP Z is a GGC then analogues of Theorem 3.1 and Proposition 3.6 have exactly the same form. However, in contrast to the OU- Γ case, one still does not have an obvious way to sample from the distribution of \mathbf{S} .

3.4 Bayesian estimation and related comments

We have shown that the distribution of $(\tau_{\theta,1}, \dots, \tau_{\theta,n})$ is determined by $2n + 2$ independent random variables whose distributions can be perfectly sampled or in the case of $v_\theta(0)$ approximated with arbitrary accuracy. We also note that the explicit densities that we have given for $M_{\theta a}$ definitely have practical utility, whereby rejection methods can be used. We also believe they are interesting from a mathematical point of view as they may have connections to application in physics or analytic combinatorics. These are places where the dilogarithm function appears often. However, in terms of practical simplicity it is perhaps easier to use the perfect simulation schemes which work for all values of θa and only require simulation from beta random variables and the distribution F_a . Also, in regards to $v_\theta(0)$, we note again that in the case of not strictly stationary OU- Γ models, we may choose $v_\theta(0)$ to have any distribution. However Theorem 3.1 suggests there are some quite interesting simplifications that occur if we choose $v_\theta(0) = T_\theta W$, where W denotes a random variable independent of T_θ . We note again that all GGC random variables have this form including the class of GIG models.

Armed with the information that we have provided one can construct a variety of efficient simulation based techniques. Here we briefly highlight the Bayesian approach. Primarily this is due to the fact that a Bayesian approach is essentially an approach involving integration and hence is a quite natural for Monte-Carlo based estimation. It is in many respects quite similar to Bootstrap techniques. We now mention some well known points about Bayesian estimation. Suppose that $\pi(\vartheta)$ is a prior distribution of the unknown parameters. Then, as is well known, the fundamental object of interest is to obtain the posterior distribution of $\vartheta|\mathbf{X}$, which is given by

$$\pi(\vartheta|\mathbf{X}) \propto \pi(\vartheta)\mathcal{L}(\mathbf{X}|\vartheta)$$

Estimation of some parameter $h(\vartheta)$ can then be cast in terms of integration,

$$(27) \quad E[h(\vartheta)|\mathbf{X}] = \int_{\Theta} h(u)\pi(du|\mathbf{X}) = \frac{\mathbb{E} \left[h(\vartheta) e^{n\bar{A}\beta} \frac{2K_\nu(\delta\gamma)}{(\gamma/\delta)^\nu \Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_i^{-1/2} \right]}{\mathbb{E} \left[e^{n\bar{A}\beta} \frac{2K_\nu(\delta\gamma)}{(\gamma/\delta)^\nu \Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_i^{-1/2} \right]}$$

where the denominator should be understood as,

$$\mathcal{L}(\mathbf{X}) = \int_{\Theta} \mathbb{E}_{\vartheta} \left[e^{n\bar{A}\beta} \frac{2K_\nu(\delta\gamma)}{(\gamma/\delta)^\nu \Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_i^{-1/2} \right] \pi(d\vartheta).$$

For instance, the posterior probability that ϑ is in some region B can be evaluated by choosing $h(x) = I\{x \in B\}$. Since Bessel functions, such as $K_\nu(x)$, are available in standard mathematical computer packages, one can just draw from the joint distribution of (ϑ, \mathbf{S}) , which is readily available

from our results. That is for $l = 1, \dots, B$ draw iid random vectors $(\vartheta_l, S_{1,l}, \dots, S_{n,l})$ then (27) is approximated by

$$(28) \quad \frac{\sum_{l=1}^B h(\vartheta_l) e^{n\bar{A}\beta_l} \frac{2K_{\nu_l}(\delta_l \gamma_l)}{(\gamma_l/\delta_l)^{\nu_l} \Gamma(\kappa_l)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_{i,l}^{-1/2}}{\sum_{l=1}^B e^{n\bar{A}\beta_l} \frac{2K_{\nu_l}(\delta_l \gamma_l)}{(\gamma_l/\delta_l)^{\nu_l} \Gamma(\kappa_l)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_{i,l}^{-1/2}}.$$

The nice feature of basic iid Monte-Carlo type estimator like (28) is that accuracy issues are well-understood and are less dependent on the sample size. Here accuracy increases as B increases.

One can of course develop more sophisticated importance sampling and MCMC methods based on well-known ideas. These may involve sampling from the posterior distributions. For instance, our results show that the posterior distribution of $\vartheta|\mathbf{X}$ can be obtained by working with the posterior distributions of $\vartheta|\mathbf{X}, \mathbf{S}, V$ and $\mathbf{S}, V|\mathbf{X}, \vartheta$ where for instance $V|\mathbf{S}, \mathbf{X}, \vartheta$ has a $GIG(\nu, \delta, \gamma)$ distribution. All other conditionals can be easily deduced by various augmentations of the expressions given in Theorem 3.1 and Proposition 3.6.

3.5 OU- Γ processes with possibly random scale parameter

Up to this point we have assumed that G_θ was a homogeneous Gamma process with scale parameter equal to 1. This was done mainly for notational convenience. However, it follows from our analysis that the introduction of a scale parameter say ζ can be used as a powerful modeling tool. Naturally a scale parameter can just be introduced by replacing G_θ with ζG_θ throughout. However an important fact is that if we use ζG_θ , the vector \mathbf{S} described in section 3.3 still does not depend on ζ . This means that one can now write

$$X_i = \mu\Delta + \beta\zeta T_{\theta(1+na)} S_i + [\zeta T_{\theta(1+na)} S_i]^{1/2} \epsilon_i.$$

Note that if ζ is fixed then all our results carry over without change. This means extending the model to the case where ζ is random is straightforward. The main feature being that we would now be working with a Gamma scale mixture, based on $\zeta T_{\theta(1+na)}$, which can be used to introduce more distributional modeling flexibility.

3.6 Likelihoods for Superpositioned OU- Γ

BNS (2001a, p.178) propose the idea of superpositions of independent OU processes to alter the auto-correlation structure. Here, letting p denote a positive integer, and (w_1, \dots, w_p) a possibly unknown vector of positive terms summing to 1, we discuss briefly a generalization of Theorem 3.1 to the case where one starts with a superposition process $v(t|p) = \sum_{j=1}^p w_j v_{\theta_j}(t)$ where for $j = 1, \dots, p$, $v_{\theta_j}(t)$ are independent OU- Γ processes which are based on parameters (λ_j, θ_j) , in place of (λ, θ) . Obviously the distributional results we have developed apply to each of the independent components. One uses for instance $a_j = \lambda_j \Delta$ and $\theta_j a_j$ in place of a and θa .

Let $\tau(t|p) = \sum_{j=1}^p w_j \tau_{\theta_j}(t)$, denote the integrated volatility where each $\tau_{\theta_j}(t) = \int_0^t v_{\theta_j}(s) ds$. Additionally the analog of $(\tau_{\theta,1}, \dots, \tau_{\theta,n})$ is $\tau_i := \tau(i\Delta) - \tau((i-1)\Delta)$. Then by similar arguments to the previous section one can write for $\xi_n = \sum_{j=1}^p \theta_j (1 + n\lambda_j \Delta)$,

$$\tau_i = T_{\xi_n} S_{i,p}$$

where $S_{i,p} := \tau_i / T_{\xi_n}$ has an obvious description by applying our previous results to each component τ_{θ_j} , and the vector $(S_{1,p}, \dots, S_{n,p})$ is independent of T_{ξ_n} .

Proposition 3.7 *Let $X_i = \mu\Delta + \beta T_{\xi_n} S_{i,p} + [T_{\xi_n} S_{i,p}]^{1/2} \epsilon_i$. with terms defined in this section. Set $\gamma^2 = [2 + \beta^2 \sum_{j=1}^n S_{i,p}]$ and $\delta^2 = \sum_{j=1}^n A_{j,p}^2 / (2S_{j,p})$, $\kappa = \xi_n = \sum_{j=1}^p \theta_j (1 + n\lambda_j \Delta)$ and $\nu = \kappa - n/2$. Let ϑ_p denote the enlarged parameter space containing unknown quantities such as (w_1, \dots, w_p) , then the likelihood $\mathcal{L}(\mathbf{X}|\vartheta_p)$ has the same form as the likelihood in Theorem 3.1 with appropriate substitutions of the above parameters and $(S_{1,p}, \dots, S_{n,p})$ in place of (S_1, \dots, S_n) . In particular,*

(i) if $[\sum_{j=1}^p \theta_j \lambda_j] \Delta = 1/2$ and $\sum_{j=1}^p \theta_j = m + 1/2$ for $m = 0, 1, 2, \dots$, then

$$\mathcal{L}(\mathbf{X}|\vartheta_p) = e^{n\bar{A}\beta} \sum_{k=0}^m \frac{(m+k)!}{k!(m-k)!2^k} \mathbb{E} \left[e^{-\delta\gamma} \frac{2(\delta\gamma)^{-k} \gamma^{-1}}{(\gamma/\delta)^m \Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_{i,p}^{-1/2} \right] \sqrt{\frac{\pi}{2}}.$$

This expression holds more generally for $\nu = m + 1/2$ or $\nu = -m - 1/2$.

(ii) If additionally $m = 0$, that is $\sum_{j=1}^p \theta_j = 1/2$ and $[\sum_{j=1}^p \theta_j \lambda_j] \Delta = 1/2$, then

$$\mathcal{L}(\mathbf{X}|\vartheta_p) = e^{n\bar{A}\beta} \mathbb{E} \left[e^{-\delta\gamma} \frac{2\gamma^{-1}}{\Gamma(\kappa)} \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} S_{i,p}^{-1/2} \right] \sqrt{\frac{\pi}{2}}.$$

□

REMARK 11. Note that superpositioning allows more flexibility in terms of the parameter values for the constraints $\sum_{j=1}^p \theta_j = m + 1/2$ and $[\sum_{j=1}^p \theta_j \lambda_j] \Delta = 1/2$. But otherwise preserves the simplicity of the likelihood as seen in (i) and (ii) of Proposition 3.7.

3.7 Randomly sampled times

From a practical point of view it may be desirable to sample at uneven or random intervals. See for instance Ait-Sahalia and Mykland (2003, 2004). The next result shows that the independence structure still holds (conditionally) but that the individual terms are not identically distributed.

Proposition 3.8 *Let $0 = \gamma_0 < \gamma_1 < \gamma_2 < \dots < \gamma_n$ denote n random times and define $\Delta_i := \gamma_i - \gamma_{i-1}$. Define $\tau_{\theta,i} := \tau_{\theta}(\gamma_i) - \tau_{\theta}(\gamma_{i-1})$, and $r_i = e^{\lambda\gamma_i}$ for $i = 1, \dots, n$, with $r_0 = 0$. Then it follows that, conditional on $(\Delta_1, \dots, \Delta_n)$, for $i = 1, \dots, n$,*

$$\lambda\tau_{\theta,i} = (1 - e^{-\lambda\Delta_i})e^{-\lambda\gamma_{i-1}}[v_{\theta}(0) + \sum_{j=1}^{i-1} r_j T_j M_j] + T_i[1 - M_i]$$

where (T_i, M_i) are conditionally independent pairs independent of $v_{\theta}(0)$. Additionally, for each fixed i , T_i and M_i are independent with distributions specified by $T_i \stackrel{d}{=} T_{\theta\lambda\Delta_i}$ and $M_i \stackrel{d}{=} M_{(\theta\lambda\Delta_i)F_{\lambda\Delta_i}}$. If the Δ_i for $i = 1, \dots, n$ are independent then the unconditional distribution of the pairs (T_i, M_i) are independent.

3.7.1 Time changed Integrated OU- Γ processes

Notice that the previous proposition places minimal constraints on the possibly random times (γ_i) . Naturally if one can easily sample $(\Delta_1, \dots, \Delta_n)$, then this would lead to models which are amenable to likelihood estimation. These observations lead us to introduce briefly a class of time changed integrated OU processes defined as

$$(29) \quad \tau_{\theta}(Z(t)) = \int_0^{Z(t)} v(s)ds = \lambda^{-1}[(1 - e^{-\lambda Z(t)})v_{\theta}(0) + \int_0^{Z(t)} (1 - e^{-\lambda(Z(t)-y)})G_{\theta}(d\lambda y)]$$

where Z is any subordinator independent of G_{θ} . The next result shows how this model is represented by Proposition 3.8.

Proposition 3.9 *Consider $\tau_{\theta}(Z(t))$ defined as in (29). For $i = 1, \dots, n$, define $\tau_{\theta,i,Z} := \tau_{\theta}(Z(i\Delta)) - \tau_{\theta}(Z((i-1)\Delta))$. Then it follows that $\tau_{\theta,i,Z}$ is equivalent to a specific $\tau_{\theta,i}$ in Proposition 3.8 by setting $\gamma_i = Z(i\Delta)$. Furthermore $\Delta_i = Z(i\Delta) - Z((i-1)\Delta) \stackrel{d}{=} Z(\Delta)$ are iid. □*

REMARK 12. The time changed process (29) represents an extremely rich class of models which adds a great deal of distributional flexibility to the OU- Γ models. As seen from Proposition 3.9 likelihood analysis for such models is again easily accomplished. In that case there may be additional unknown parameters associated with Z . For instance selecting Z such that

$$\mathbb{E}[e^{-\omega Z(\Delta)}] = e^{-\Delta[(b+\omega)^{1/2} - b^{1/2}]}$$

Corresponds to the case where $Z(\Delta)$ is an Inverse Gaussian random variable.

REMARK 13. One may also replace $Z(t)$ in (29) with any tractable increasing process. For instance one may choose $\tau_\alpha^*(t)$ to be an integrated OU- Γ process independent of τ_θ

REMARK 14. Leverage type models discussed in BNS pose no extra difficulties. In the simplest likelihood setting, this translates into replacing $X_i = \mu\Delta + \beta\tau_i + \tau_i^{1/2}\epsilon_i$ described in (6), with

$$X_i = \mu\Delta + vT_i + \beta\tau_i + \tau_i^{1/2}\epsilon_i.$$

Where $T_i \stackrel{d}{=} T_{\theta a}$ and τ_i is otherwise related to T_i by the representation given in Proposition 3.5. v is a real-valued unknown quantity.

REMARK 15. We can extend the OU- Γ processes based on the homogeneous process G_θ to one based on an inhomogeneous Gamma process $G_{\theta\nu}$, where ν is an appropriately defined sigma-finite measure. That is the Lévy exponent for any positive function g of $G_{\theta\nu}(g) = \int_0^\infty g(x)G_{\theta\nu}(dx)$ is given by $\int_0^\infty d_\theta(\omega g(x))\nu(dx)$. The volatility process is then defined by

$$v_{\theta\nu}(t) = e^{-\lambda t}v(0) + e^{-\lambda t} \int_0^t e^{\lambda y} G_{\theta\lambda\nu}(dy)$$

The process $v_{\theta\nu}(t)$ is stationary only in the homogeneous case. However the independence properties that we exploited still hold and one has fairly obvious generalizations of the results we have presented. An advantage is that this is another way to increase distributional flexibility.

3.8 The special nature of the OU- Γ process as an SV model

It is important to note that this independence phenomena, exhibited in Proposition 3.1, which allows one to easily describe the joint structure of (τ_1, \dots, τ_n) for a potential SV model is not only due to the usage of a Gamma process G_θ . That is to say it will not necessarily be true for non-OU models based on G_θ . To see this define a moving average process of the type

$$\int_0^t (t-x)e^{-(t-x)} G_\theta(dx)$$

It is not difficult to see that the analog of (7) amounts to $(\int_0^a e^{-y} G_\theta(dy), \int_0^a ye^{-y} G_\theta(dy))$. To see the problem first set H_a to be uniform $[0, a]$, and $g_1(y) = e^{-y}$, and $g_2(y) = ye^{-y}$. Then it clear that the pair above are equivalent in distribution to the pair

$$(30) \quad (T_{\theta a} P_{\theta a H_a}(g_1), T_{\theta a} P_{\theta a H_a}(g_2))$$

The good point about this representation is that the marginal distributional results for Dirichlet process mean functionals apply. This means, for instance, that basically all Lévy moving average processes that are driven by a Z which is an FGGC have the property that any calculation involving a one-dimensional random variable can be calculated using the marginal distributional results for Dirichlet process mean functionals. This has an immediate consequence for option pricing formula based on such models.

However it is quite clear from (30) that one can negotiate the dependence structure in a manner similar to Proposition 3.1, if and only if $P_{\theta_a H_a}(g_1)$ can be expressed as a function of $P_{\theta_a H_a}(g_2)$, which is not true for this example. This is also why in (7) the OU-FGGC models we shall discuss do not have the structure exhibited in Proposition 3.1. In other words $Z(a)$ in that expression has to have a Gamma distribution. Or more generally expressible as T_{θ_a} and a function of the other coordinate. Of course the OU- Γ is not the only Gamma driven SV model that has the ability to be exactly sampled as we did in this section. Another example is the Dykstra and Laud (1981) type model, see also James (2005b, p. 1784, eq. (29)), which takes the simple form

$$\int_0^t (t-x)G_\theta(dx).$$

In this case the analogue of (7) amounts to $(G_\theta(a), \int_0^a yG_\theta(dy))$.

4 General Likelihoods

We now proceed to show how one may perform likelihood analysis for more general (τ_1, \dots, τ_n)

4.1 Fourier-Cosine integral representation of the likelihood

In order to calculate (10) we use the classical Fourier-Cosine integral

$$(31) \quad \frac{1}{\pi} \int_0^\infty \cos(y|A_i|) e^{-\frac{y^2 \tau_i}{2}} dy = \frac{1}{\sqrt{2\pi}} \tau_i^{-1/2} e^{-\frac{A_i^2}{2\tau_i}}.$$

This is a special of the Bessel integral identities known as Weber-Sonine formula. See for instance Andrews, Askey and Roy (1999, p.222) and Watson (1966, p. 394 eq. (4)) for the identity and also those references for Bessel functions. It now follows rather immediately that,

Proposition 4.1 *For the model described by (6), let (τ_1, \dots, τ_n) have an arbitrary distribution where the joint Laplace transform has a known form. Then the marginal likelihood is given by,*

$$\mathcal{L}(\mathbf{X}|\vartheta) = \frac{e^{n\bar{A}\beta}}{\pi^n} \int_{\mathbb{R}_+^n} \mathbb{E} \left[\prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} \right] \prod_{i=1}^n \cos(y_i|A_i|) dy_i$$

where

$$\mathbb{E} \left[\prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} \right]$$

is the joint Laplace transform of (τ_1, \dots, τ_n) evaluated at $\omega_i = y_i^2/2 + \beta^2/2$ for $i = 1, \dots, n$. \square

The next result which first appears in James (2005c)[see also James (2005b)], which can be thought of an unpublished earlier version of this manuscript, describes the case where τ_i is representable as a functional of a Poisson random measure. Since positive Lévy processes can be constructed from Poisson random measures this represents a very rich class.

Proposition 4.2 *Let N denote a Poisson random measure on a Polish space \mathcal{X} with sigma-finite mean intensity ν , such that for each positive function g , the corresponding random variable $N(g)$ has Lévy exponent $\Psi(\omega g) = \int_{\mathcal{X}} (1 - e^{-g(x)\omega}) \nu(dx)$. Suppose that $\tau_i = N(g_i)$ for positive functions (g_i) on \mathcal{X} . Then since $\sum_{i=1}^n N(\omega_i g_i) = N(\sum_{i=1}^n \omega_i g_i)$, it follows that (τ_1, \dots, τ_n) has the joint Lévy exponent $\Psi(\Omega) = \Psi(\sum_{i=1}^n \omega_i g_i)$. Then for the model described by (6), the likelihood is given by,*

$$\mathcal{L}(\mathbf{X}|\vartheta) = \frac{e^{n\bar{A}\beta}}{\pi^n} \int_{\mathbb{R}_+^n} e^{-\Psi(\Omega)} \prod_{i=1}^n \cos(y_i|A_i|) dy_i$$

where $\Omega(x) = \sum_{i=1}^n \omega_i g_i(x)$ with $\omega_i = y_i^2/2 + \beta^2/2$ for $i = 1, \dots, n$. \square

REMARK 16. Notice that we have stated the result in terms of quite arbitrary (τ_1, \dots, τ_n) . This is because the expression (31) has nothing to do with the distributional properties of τ .

REMARK 17. Hereafter we set

$$(32) \quad \mathcal{C}(\mathbf{y}|\mu) = \prod_{i=1}^n \cos(y_i|A_i)$$

REMARK 18. The appearance of integrals involving Bessel functions is certainly not new to applications in finance as can be seen in the case of the important work of Yor (1992) on Asian Options. See also Carr and Schröder (2004).

5 General OU likelihoods

We now apply Proposition 4.1, in the case of where (τ_1, \dots, τ_n) are based on the integrated OU models described by (11). The task is to calculate the joint Laplace transform evaluated at $(\omega_1, \dots, \omega_n)$. This is straightforward from the construction given section 2.2. which implies that

$$\lambda \sum_{i=1}^n \omega_i \tau_i = s_1 v(0) + \sum_{l=1}^{n-1} [s_{l+1} O_l r_l + [Z_l - O_l] \omega_l] + [Z_n - O_n] \omega_n$$

where for $l = 1, \dots, n$, $s_l = (1 - e^{-\lambda \Delta}) [\sum_{i=l}^n \omega_i e^{-\lambda(i-1)\Delta}]$. Then it is not difficult to see that the joint Laplace transform of (τ_1, \dots, τ_n) is of the form

$$(33) \quad L_1(\mathbf{y}|\vartheta) = e^{-\varphi(s_1)} e^{-\Phi(\omega_n)} \left[\prod_{i=1}^{n-1} e^{-\Phi(\omega_i|v_i)} \right]$$

where terms are explicitly defined in the next result which gives the likelihood.

Proposition 5.1 *For the model described by (6), let (τ_1, \dots, τ_n) be defined by the OU models as in (11). Then the marginal likelihood in (10) is,*

$$\mathcal{L}(\mathbf{X}|\vartheta) = \frac{e^{n\bar{A}\beta}}{\pi^n} \int_{\mathbb{R}_+^n} L_1(\mathbf{y}|\vartheta) \mathcal{C}(\mathbf{y}|\mu) \prod_{i=1}^n dy_i,$$

where $\omega_i = y_i^2/2 + \beta^2/2$ and $v_i = r_i s_{i+1}$. $\mathcal{C}(\mathbf{y}|\mu)$ is defined in (32) and $L_1(\mathbf{y}|\vartheta)$ is the joint Laplace transform evaluated at $(\omega_1, \dots, \omega_n)$ with form specified by (33). The Lévy exponents in (33) are specifically defined as follows, for $a = \lambda\Delta$,

$$(i) \quad \Phi(\omega_i|v_i) = \int_{e^{-a}}^1 \psi(\lambda^{-1}[v_i u + \omega_i(1-u)]) \frac{du}{u}, \text{ for } i = 1, \dots, n-1$$

$$(iii) \quad \Phi(\omega_n) = \int_{e^{-a}}^1 \psi(\lambda^{-1}\omega_n(1-u)) \frac{du}{u}$$

$$(iv) \quad \varphi(s_1) = \int_0^1 \psi(s_1 \lambda^{-1} u) \frac{du}{u}, \text{ is the Lévy exponent of } v(0) \text{ evaluated at } s_1 \lambda^{-1}.$$

□

Consider now the following result which we will return to in section 6.

Proposition 5.2 *Consider $\Phi(\omega_i|v_i)$, $\Phi(\omega_i)$, and let $\Lambda(v_i|\omega_i) = \Phi(\omega_i|v_i) - \Phi(\omega_i)$. Define*

$$D_\rho(y, e^a y|\omega_i) = \int_y^\infty e^{-\omega_i s} \rho(ds) - \int_y^\infty e^{-\omega_i s} \rho(ds)$$

Then

$$(i) \quad \Lambda(v_i|\omega_i) = a \int_0^1 \int_0^\infty (1 - e^{-v_i u s}) e^{-\omega_i(1-u)s} \rho(ds) F_a(du).$$

$$(ii) \quad \Lambda(v_i|\omega_i) = \int_0^\infty (1 - e^{-v_i y}) D_\rho(y, e^a y | w_i) y^{-1} e^{-y} e^{w_i y} dy.$$

(iii) It follows that for each fixed ω_i , $\Lambda(t|\omega_i)$ is the Lévy exponent, evaluated at t , of an infinitely divisible random variable with Lévy density $D_\rho(y, e^a y | w_i) y^{-1} e^{-y} e^{w_i y}$.

PROOF. Note that $\Lambda(v_i|\omega_i) = a \int_{e^{-a}}^1 [\psi([v_i u + \omega_i(1-u)]) - \psi([\omega_i(1-u)])] F_a(du)$. Statement (i) is simply the Lévy density representation of this. Statement (ii) follows by the change of variable $y = us$, and exploiting the scale invariance of the measure $u^{-1} du$. \square

This allows one to better understand the representation of the joint Laplace transform

$$(34) \quad L_1(\mathbf{y}|\vartheta) = e^{-\varphi(s_1)} e^{-\Phi(\omega_n)} \left[\prod_{i=1}^{n-1} e^{-\Phi(\omega_i|v_i)} \right] = e^{-\varphi(s_1)} \left[\prod_{i=1}^n e^{-\Phi(\omega_i)} \right] \prod_{i=1}^{n-1} e^{-\Lambda(v_i|\omega_i)}$$

where ω_i depends only on y_i and each v_i depends on (y_{i+1}, \dots, y_n) . We also note that

$$(35) \quad L_2(\mathbf{y}|\vartheta) := e^{-\varphi(s_1)} \left[\prod_{i=1}^n e^{-\Phi(\omega_i)} \right]$$

is also a joint Laplace transform. In fact examining (35) more closely we see that it is the joint Laplace of a sequence random variables (ξ_1, \dots, ξ_n) , where $\lambda \xi_i = c_i v(0) + [Z_i - O_i]$. Here $c_i = (1 - e^{-a}) e^{-a(i-1)}$. However note that if $c_i = (1 - e^{-a})$, then when $v(t)$ is stationary, it follows that the marginal distribution of this version of ξ_i is equivalent to τ_i . Since we later propose the use of joint densities based on (34) and (35) one may want to vary the value of c_i in (35) as this may increase accuracy.

6 Some distribution theory for OU-FGGC models

We have already mentioned that the class of infinitely divisible random variables which are GGC's are closely linked with Dirichlet process mean functionals. When the Gamma process has a finite shape measure say θH , then every such GGC can be expressed as $T_\theta M_{\theta H}$ where $M_{\theta H} = \int_0^\infty x P_{\theta H}(dx)$ is a Dirichlet process. We will call such GGC's *finite* GGC's or FGGC. One implication is that one may apply some of the distribution theory we have developed for the OU-Γ to these models. In this section we shall assume that Z is derived from a finite GGC and demonstrate some nice properties of the corresponding OU process which are also relevant to sampling likelihoods and option pricing calculations. First note that if Z is an FGGC then its Lévy density and Lévy exponent are given by

$$(36) \quad \psi(\omega) = \int_0^\infty d_\theta(\omega x) H(dx) \quad \text{and} \quad \theta y^{-1} \int_0^\infty e^{-y/r} H(dr)$$

where H is a probability measure

REMARK 19. The term *finite* GGC should not be confused with the term *finite activity*. That is to say FGGC are *infinite activity* models as can be seen from their Lévy density in (36).

6.1 Results for perfect sampling relevant OU-FGGC components

Proposition 6.1 Suppose that Z is a BDLP with specifications given in (36). Then the following results hold.

- (i) In the stationary case the corresponding OU process $v(t)$ is such that $v(0)$ is a non-finite GGC with Lévy exponent

$$\int_0^\infty d_\theta(\omega u) S(u) u^{-1} du = \int_0^1 \int_0^\infty d_\theta(\omega u x) H(dx) u^{-1} du$$

where $S(u) = \int_u^\infty H(dy)$ is a survival function.

- (ii) Consider the Lévy exponent $\Phi(\omega)$ described in Proposition 4.1. Then in this setting it takes the form

$$(37) \quad \Phi(\omega) = \int_0^\infty d_{\theta a}(\omega r) Q_a(dr) = \int_{e^{-a}}^1 \int_0^\infty d_{\theta a}(\omega(1-u)x) H(dx) F_a(du)$$

where Q_a is a probability measure corresponding to the distribution of a random variable $R = (1-U)W$ where U has distribution F_a and W is independent of U and has distribution H .

- (iii) Equivalently $Z(a) - Y_{\theta a} \stackrel{d}{=} \int_0^\Delta (1 - e^{-\lambda(\Delta-y)}) Z(d\lambda y)$ is a random variable with Lévy exponent (37) and hence is a finite GGC and has the representation $Z(a) - Y_{\theta a} \stackrel{d}{=} T_{\theta a} M_{\theta a Q_a}$
- (iv) If the support of H is finite then the support of Q_a is finite. This implies that the perfect simulation method described by (17) and (18) applies to $M_{\theta a Q_a}$. One may choose $u M_{\theta a Q_a, -N}$ and $l M_{\theta a Q_a, -N}$, according to the upper and lower support points of Q_a . $B_{n, \theta a}$ is BETA $(1, \theta a)$ and $X_n \stackrel{d}{=} R = W(1-U)$ has distribution Q_a .
- (v) For $\theta a = 1$ the density of M_{Q_a} has the form,

$$f_{M_{Q_a}}(x) = \frac{1}{\pi} \sin(\pi Q_a(x)) e^{-\int_0^\infty \log(|t-x|) Q_a(dt)}$$

where $Q_a(x) = \int_0^x Q_a(dt)$. Densities for $\theta a > 1$ are obtained by substituting $\theta a = \theta$ and $Q_a = H$ in (16).

- (vi) The Levy exponent of $Y_{\theta a} \stackrel{d}{=} \int_0^\Delta e^{-\lambda(\Delta-y)} Z(d\lambda y)$ is similar to (37) but with $d_{\theta a}(\omega u x)$ in place of $d_{\theta a}(\omega(1-u)x)$. Hence $Y_{\theta a} \stackrel{d}{=} T_{\theta a} M_{\theta a \tilde{Q}_a}$, where \tilde{Q}_a corresponds to the distribution of $\tilde{R} = WU$.
- (vi) $Y_{\theta a}$ converges in distribution to $v_\theta(0)$ as $e^{-a} \rightarrow 0$.

□

PROOF. The proof of (i) and (ii) are obvious by substituting the form of ψ in (36) into (8) and (9). The remaining results follow as consequences. The density in (v) is obtained from Cifarelli and Regazzini (1990) or Cifarelli and Melilli (2001). □

The next result is rather curious but as we shall show can play a powerful role in Monte Carlo procedures.

Proposition 6.2 Consider the setting in Proposition 6.1 then the Lévy exponent $\Lambda(t|\omega_i)$ described in Proposition 5.2 takes the form

$$\Lambda(t|\omega_i) = \int_0^\infty d_{\theta a}(tr) Q_{a|\omega_i}(dr).$$

where $Q_{a|\omega_i}$ is a probability measure corresponding to a random variable

$$R = \frac{UW}{1 + W(1 - U)\omega_i}$$

where U has distribution F_a and W has distribution H . As a consequence results analogous to Proposition 6.1 apply to this setting.

PROOF. Similar to Proposition 5.2 we examine

$$\Lambda(v_i|\omega_i) = a \int_{e^{-a}}^1 [\psi([v_i u + \omega_i(1 - u)]) - \psi([\omega_i(1 - u)])] F_a(du).$$

However in this case $\psi([v_i u + \omega_i(1 - u)]) - \psi([\omega_i(1 - u)])$ is equivalent to

$$\int_0^\infty [d_\theta([v_i u + \omega_i(1 - u)]y) - d_\theta([\omega_i(1 - u)]y)] H(dy)$$

Now using properties of the natural logarithm it follows that

$$d_\theta([v_i u + \omega_i(1 - u)]y) - d_\theta([\omega_i(1 - u)]y) = d_\theta\left(\frac{v_i u y}{1 + \omega_i(1 - u)y}\right)$$

concluding the result. \square

6.2 OU-FGGC Monte Carlo Densities

The next result, whose present importance is that it can be used effectively in Monte Carlo simulation procedures, follows immediately from Propositions 6.1 and 6.2 and standard augmentation arguments.

Theorem 6.1 *Suppose that the joint Laplace transforms $L_1(\mathbf{y}|\vartheta)$, and hence $L_2(\mathbf{y}|\vartheta)$, satisfies the conditions in Proposition 6.1 and 6.2. Specify $\omega_i = (y_i^2 + \beta^2)/2$. From this, for $i = 1, \dots, n$, we can let $(T_i, M_i) \stackrel{d}{=} (T_{\theta a}, T_{\theta a} M_{\theta a Q_a})$ denote iid pairs of random variables. Similarly, independent of the above sequence, define independent pairs (G_i, M_{ω_i}) , where $G_i \stackrel{d}{=} T_{\theta a}$ and independent of G_i , M_{ω_i} has the distribution of a mean functional described in Proposition 6.2 for fixed ω_i . Let*

$$\Xi_1 = (v(0), (T_i, M_i), (G_i, M_{\omega_i}))$$

denote the joint vector of $4n + 1$ independent components, with joint density $f_{\Xi_1}(\cdot|\vartheta, \mathbf{y})$. Similarly let $\Xi_2 = (v(0), (T_i, M_i))$ denote the joint vector of $2n + 1$ independent components with density $f_{\Xi_2}(\cdot|\vartheta)$ specified by Proposition 6.1 and not depending on \mathbf{y} . Then,

- (i) $L_1(\mathbf{y}|\vartheta) = \mathbb{E}[e^{-s_1 v(0)}] \prod_{i=1}^n \mathbb{E}[e^{-\omega_i T_i M_i}] \prod_{i=1}^n \mathbb{E}[e^{-v_i G_i M_{\omega_i}}]$
- (ii) $L_2(\mathbf{y}|\vartheta) = \mathbb{E}[e^{-s_1 v(0)}] \prod_{i=1}^n \mathbb{E}[e^{-\omega_i T_i M_i}]$
- (iii) Suppose that $\int_{\mathbb{R}_+^n} L_1(\mathbf{y}|\vartheta) \prod_{i=1}^n dy_i < \infty$, then by augmenting the expression in (i) there exists a joint density of (Ξ_1, \mathbf{Y}) given by

$$(38) \quad f_{\Xi_1}(\zeta_1, \mathbf{y}|\vartheta) \propto e^{-s_1 v} \prod_{i=1}^n e^{-\omega_i t_i m_i} \prod_{i=1}^n e^{-v_i u_i m_{\omega_i}} f_{\Xi_1}(\zeta_1|\vartheta, \mathbf{y})$$

where $\zeta_1 = (v, (t_i, m_i), (u_i, m_{\omega_i}))$, with obvious meaning.

(iv) Suppose that $\int_{\mathbb{R}_+^n} L_2(\mathbf{y}|\vartheta) \prod_{i=1}^n dy_i < \infty$, then by augmenting the expression in (ii) there exists a joint density of (Ξ_2, \mathbf{Y}) given by

$$f_{\Xi_2}(\zeta_2, \mathbf{y}|\vartheta) \propto e^{-s_1 v} \prod_{i=1}^n e^{-\omega_i t_i m_i} f_{\Xi_2}(\zeta_2|\vartheta, \mathbf{y}),$$

where $\zeta_2 = (v, (t_i, m_i))$.

(v) Writing $v(0) = T_\theta \tilde{M}_\theta$ and integrating out all the Gamma random variables in (iii) it follows that there exist a joint density of $(\tilde{M}_\theta, (M_i), (M_{\omega_i}), \mathbf{Y})$ given proportional to

$$(39) \quad f_{\tilde{M}_\theta}(t)(1 + s_1 t)^{-\theta} \prod_{i=1}^n (1 + \omega_i m_i)^{-\theta a} (1 + v_i r_i)^{-\theta a} f_{M_i}(m_i) f_{M_{\omega_i}}(r_i).$$

6.3 OU-FGGC option pricing densities

The last result, extends Proposition 3.4 and again is pertinent to the option pricing formula discussed in BNS(2001a, 6.2) and Nicolato and Vernados (2003).

Proposition 6.3 Let $x^*(t)$ be defined by the BDLP Z which is an FGGC with specifications (36). Additionally, for $0 \leq s < t$, set $\Delta = (t - s)$ and define $h(\Delta, s) = (1 - e^{-\lambda \Delta})v(s)$ and $\mu_s^* = \mu \Delta + x^*(s) + \beta h(\Delta, s)$. Then the conditional density of $x^*(t)|x^*(s), v(s)$ is given by

$$\int_0^\infty \phi(x|\mu_s^* + \beta y, h(\Delta, s) + y) q_{\theta a}(y) dy$$

where $q_{\theta a}(y) = \int_0^\infty \mathcal{G}_{\theta a}(y|v) f_{M_{\theta a} Q_a}(v) dv$. With the density further described by the specifications in Proposition 6.1 \square

7 Some practical issues for general OU likelihood estimation

The likelihoods given in Propositions 4.1, 4.2 and 5.1 serve the purpose of integrating out the infinite-dimensional nuisance parameters. A natural question is how to exploit these results in a practical sense. In the forthcoming sections we shall focus on the OU models but many parts of our discussion can be extended to more general processes where the joint Laplace transform has an accessible form. Our goal at minimum will be to discuss ways to evaluate the likelihood by Monte Carlo procedures. This could then be used in conjunction with simulated maximum likelihood estimation or other such techniques. Similar to section 3.4 we will also be thinking about Bayesian type estimation procedures. That is, we wish to calculate

$$(40) \quad \mathbb{E}[h(\vartheta)|\mathbf{X}] = \frac{\int_{\mathcal{S}} h(\vartheta) \pi(d\vartheta) e^{n\bar{A}\beta} L_1(\mathbf{y}|\vartheta) \mathcal{C}(\mathbf{y}|\mu) \prod_{i=1}^n dy_i}{\int_{\mathcal{S}} \pi(d\vartheta) e^{n\bar{A}\beta} L_1(\mathbf{y}|\vartheta) \mathcal{C}(\mathbf{y}|\mu) \prod_{i=1}^n dy_i}$$

where we set $\mathcal{S} = (\mathbb{R}_+^n, \Theta)$.

7.1 Calculating Lévy exponents

In order to utilize Proposition 5.1 one needs a manageable expression for

$$(41) \quad \Phi(\omega_1|\omega_2) = \int_{e^{-a}}^1 \psi([\omega_2 u + \omega_1(1 - u)]) \frac{du}{u} := a \int_{e^{-a}}^1 \psi([\omega_2 u + \omega_1(1 - u)]) F_a(du)$$

where ω_1 and ω_2 just denote two arbitrary non-negative numbers. We have removed the dependence on the scale factor λ^{-1} , which can otherwise be absorbed in (ω_1, ω_2) . We will assume that φ has a known form. One can see that (41) is the Lévy exponent of the joint distribution of (7). However, even if we wished to try to apply a direct inversion the results for the OU- Γ would suggest that, in general, the joint density of (7) has a rather non-obvious form. That is to say, except for the OU- Γ case, it is probably just as well to work directly with (41). Now again note importantly that in order to calculate (41) we only need knowledge of ψ and not the Lévy density ρ of Z . In many cases manual evaluation of (41) may not be obvious. One can then resort to numerical methods available in standard mathematical packages or one can carry out a one time Monte-Carlo approximation based on the following, somewhat obvious, result.

Proposition 7.1 *Let U_l for $l = 1, \dots, B$ denote iid random variables with distribution F_a . Let $B\hat{\Phi}(\omega_1|\omega_2) = \sum_{l=1}^B \psi(\omega_2 U_l + \omega_1(1 - U_l))$ Then*

$$\mathbb{E} \left[\hat{\Phi}(\omega_1|\omega_2) \right] = \Phi(\omega_1|\omega_2)$$

□

Note that our intention is to use a one time calculation of $\hat{\Phi}(\omega_1|\omega_2)$, based on large B , to get a highly accurate approximation to $\Phi(\omega_1|\omega_2)$. Our intent is not to continuously generate different realizations of $\hat{\Phi}(\omega_1|\omega_2)$ within a loop. In other words one stores a set of (U_l) . The remaining sections will assume that we have been able to get an expression for $\Phi(\omega_1|\omega_2)$ by some means.

REMARK 20. As seen from our results in section 6 we do not necessarily need to work with (41) in the case of OU-FGGC models.

7.2 Monte Carlo method

It is well-known that classical iid Monte-Carlo, MCMC and SIS procedures are well-suited to high-dimensional integrals. However, at first glance, one might think it is difficult to work with the expressions involving cosines. Specifically our likelihoods are expressed in terms of $\mathcal{C}(\mathbf{y}|\mu)$ which oscillates between positive and negative values. On the other hand, we note that

$$|\mathcal{C}(\mathbf{y}|\mu)| \leq |\cos(y_1|A_1)| \leq 1$$

for all (y_1, \dots, y_n) , which suggests that a product of cosines is not any more unstable than a single cosine. Monte Carlo procedures just require a reasonable proposal density and otherwise deal with terms such $\mathcal{C}(\mathbf{y}|\mu)$ in terms of an expectation $\mathbb{E}[h(\mathbf{Y})]$ where h depends on $\mathcal{C}(\mathbf{y}|\mu)$ and possibly other terms. Accuracy then becomes primarily a function of the number B of computer iterations. That is, in terms of B Monte Carlo replications. This is in contrast to numerical techniques which have difficulty handling high dimensions in n . See for instance Liu (2001), Chen, Shao and Ibrahim (2000) and Kong, Liu and Wong (1997).

The idea of Monte Carlo in the general setting is in principle no different than that outlined in section 3.4. Except now we will sample from densities built from $L_1(\mathbf{y}|\vartheta)$ and $L_2(\mathbf{y}|\vartheta)$. Similar to Theorem 6.1, this would be possible if its prospective normalizing constant was finite. Note one can sample these densities without explicit knowledge of the normalizing constant via MCMC methods. We now give a description of its normalizing constant.

Proposition 7.2 *Let $\xi_i = c_i v(0) + [Z_i - O_i]$, for $i = 1, \dots, n$. Then $\max(c_i v(0), [Z_i - O_i]) \leq \xi_i \leq \tau_i$, and the following results hold.*

- (i) $N_{\vartheta,1} = \int_{\mathbb{R}_+^n} L_1(\mathbf{y}|\vartheta) \prod_{i=1}^n dy_i = \pi^n \mathbb{E} \left[\prod_{i=1}^n \frac{e^{-\beta^2 \tau_i}}{\sqrt{2\pi \tau_i}} \right] \leq \pi^n \mathbb{E} \left[\prod_{i=1}^n \frac{1}{\sqrt{\tau_i}} \right]$
- (ii) $N_{\vartheta,2} = \int_{\mathbb{R}_+^n} L_2(\mathbf{y}|\vartheta) \prod_{i=1}^n dy_i := \pi^n \mathbb{E} \left[\prod_{i=1}^n \frac{e^{-\beta^2 \xi_i}}{\sqrt{2\pi \xi_i}} \right] \leq \pi^n \mathbb{E} \left[\prod_{i=1}^n \frac{1}{\sqrt{\xi_i}} \right] \quad \square$

Proposition 7.2 follows by a straightforward argument which can be seen more clearly in section 8. We see from Proposition 7.2 that the prospective normalizing constants are just based on negative moments of the random variables τ_i , ξ_i , $v(0)$ and $Z_i - O_i$ which may or may not exist. One can always ensure finiteness by adding a small positive constant to any of the random variables. For instance one uses the model based on $(\tau_1 + b, \dots, \tau_n + b)$ for a small $b > 0$. Hereafter we shall then assume that modification is made if deemed necessary. This now allows us to describe two possible densities for Monte Carlo implementation as follows

$$(42) \quad N_{\vartheta,1} Q_1(\mathbf{y}|\vartheta) = L_1(\mathbf{y}|\vartheta)$$

and

$$(43) \quad N_{\vartheta,2} Q_2(\mathbf{y}|\vartheta) = L_2(\mathbf{y}|\vartheta).$$

Naturally, from the point of view of Monte Carlo (theoretical) accuracy, $Q_1(\mathbf{y}|\vartheta)$ is the most desirable. However, $Q_2(\mathbf{y}|\vartheta)$ is in general easier to sample from. One can also adjust $Q_2(\mathbf{y}|\vartheta)$ further if necessary. Define the ratio

$$\Psi(\mathbf{y}|\vartheta) = \frac{Q_1(\mathbf{y}|\vartheta)}{Q_2(\mathbf{y}|\vartheta)} = \frac{N_{\vartheta,2}}{N_{\vartheta,1}} \prod_{i=1}^n e^{-\Lambda(v_i|\omega_i)}.$$

Proposition 7.3 *Consider the densities defined in (42) and (43) and the Bayesian posterior quantity given in (40). Additionally let $\mathbb{E}_{\vartheta,j}$ denote expectation with respect to the respective joint density of $\mathbf{Y} = (Y_1, \dots, Y_n)$, $Q_j(\mathbf{y}|\vartheta)$ for $j = 1, 2$. Define also \mathbb{E}_j to denote expectation with respect to the joint densities $\pi(\vartheta)Q_j(\mathbf{y}|\vartheta)$ for $j = 1, 2$. Then it follows that*

$$(i) \quad \mathcal{L}(\mathbf{X}|\vartheta) = N_{\vartheta,1} \frac{e^{n\bar{A}\beta}}{\pi^n} \mathbb{E}_{\vartheta,1}[\mathcal{C}(\mathbf{Y}|\mu)]$$

$$(ii) \quad \mathcal{L}(\mathbf{X}|\vartheta) = N_{\vartheta,1} \frac{e^{n\bar{A}\beta}}{\pi^n} \mathbb{E}_{\vartheta,2}[\mathcal{C}(\mathbf{Y}|\mu)\Psi(\mathbf{Y}|\vartheta)]$$

(iii) *This implies that*

$$(44) \quad \mathbb{E}[h(\vartheta)|\mathbf{X}] = \frac{\mathbb{E}_1[h(\vartheta)e^{n\bar{A}\beta}N_{\vartheta,1}\mathcal{C}(\mathbf{Y}|\mu)]}{\mathbb{E}_1[e^{n\bar{A}\beta}N_{\vartheta,1}\mathcal{C}(\mathbf{Y}|\mu)]} = \frac{\mathbb{E}_2[h(\vartheta)e^{n\bar{A}\beta}N_{\vartheta,2}\mathcal{C}(\mathbf{Y}|\mu)\Psi(\mathbf{Y}|\vartheta)]}{\mathbb{E}_2[e^{n\bar{A}\beta}N_{\vartheta,2}\mathcal{C}(\mathbf{Y}|\mu)\Psi(\mathbf{Y}|\vartheta)]} \square$$

Hence a Bayesian approach proceeds similar to section 3.4 by sampling $(Y_{1,l}, \dots, Y_{n,l}, \vartheta_l)$ for $l = 1, \dots, B$ times from either $\pi(\vartheta)Q_1(\mathbf{y}|\vartheta)$ or $\pi(\vartheta)Q_2(\mathbf{y}|\vartheta)$ and put them into appropriate empirical versions of (44). We now say a few more words about sampling from the respective densities

7.2.1 Sampling from Q_1

In general an exact expression for the conditional marginals of say $Y_k|Y_1, \dots, Y_n$ based on $Q_1(\mathbf{y}|\vartheta)$ can be worked out but it is a bit tricky. As such we do not discuss this. Note that for OU-FGGC models one can definitely use Theorem 6.1 to sample from the joint distribution of $(\Xi_1, \mathbf{Y}, \vartheta)$ based on (38) or sampling based on the density (39). These methods are facilitated by the fact that we can use the perfect simulation methods described in section 2.3.2., with specifications given by Proposition 6.1 and 6.2.

7.2.2 Sampling from Q_2

Notice that in general $Q_2(\mathbf{y}|\vartheta)$ has an almost independent structure and hence a rejection sampling procedure is straightforward. If however we know the distribution of v_0 we can introduce a further augmentation based on

$$e^{-\varphi(s_1)} = \int_0^\infty e^{-vs_1} f_{v(0)}(v) dv.$$

where again $s_1 = \sum_{i=1}^n c_i(y_i^2 + \beta^2)/2$. Hence the Monte Carlo procedure can be based on a joint density of $(\mathbf{Y}, V|\vartheta)|\vartheta$ given as

$$Q_2(\mathbf{y}, v) \propto \left[\prod_{i=1}^n e^{-y_i^2 v c_i / 2} e^{-\Phi(\omega_i)} \right] e^{-v \beta^2 \sum_{i=1}^n c_i / 2} f_{v(0)}(v).$$

In the OU-FGGC case we may again use Theorem 6.1 in an obvious way.

8 General approach

So far we have advocated the idea of sampling using the joint Laplace transform or some variation of that. Since we focused on the BNS models we were able to highlight some nice features. However our claim is that one can implement similar procedures. This leads us to derive a similar approach that is influenced by some arguments in Devroye (1986a) but where we do not necessarily sample using the Laplace transform. That is we give another representation of the likelihood that can be numerically evaluated via the simulation of random variables. First let $\mathbf{p} = (p_1, \dots, p_n)$ denote a vector of positive numbers and for each i , let

$$H(y_i|p_i) = \frac{2}{\sqrt{2\pi p_i}} e^{-\frac{y_i^2}{2p_i}} \text{ for } y_i > 0$$

denote a half Normal density. Now, notice that $0 \leq 1 - \prod_{i=1}^n \cos(y_i) \leq 2$, and

$$(45) \quad \int_{\mathbb{R}_+^n} \left[1 - \prod_{i=1}^n \cos(y_i|A_i) \right] H(y_i|p_i) dy_i = 1 - e^{-\frac{\sum_{i=1}^n A_i^2 p_i}{2}} = C_n(\mathbf{A}, \mathbf{p})$$

This follows from applications of the Fourier-Cosine identity that we used in section 4.1. From these facts we describe a joint density

Proposition 8.1 *Augmenting the expression in (45) leads to a joint density of an array of positive random variables $\mathbf{Y} = \{Y_{1,n}, \dots, Y_{n,n}\}$ given by,*

$$r_n(\mathbf{y}|\mathbf{p}) = \frac{[1 - \prod_{i=1}^n \cos(y_i|A_i)] \prod_{i=1}^n H(y_i|p_i)}{C_n(\mathbf{A}, \mathbf{p})}$$

Equivalently, for $k = 1, \dots, n$, the conditional density of $Y_{k,n}|Y_{1,n}, \dots, Y_{k-1,n}$ is proportional to $[1 - \lambda_k \cos(y_k|A_k)] H(y_k|p_k)$, where $\lambda_k = e^{-\sum_{i=k+1}^n \frac{A_i^2 p_i}{2}} \prod_{i=1}^{k-1} \cos(y_i|A_i)$ for $k = 2, \dots, n-1$, $\lambda_1 = e^{-\sum_{i=2}^n \frac{A_i^2 p_i}{2}}$, and $\lambda_n = \prod_{i=1}^{n-1} \cos(y_i|A_i)$.

Define the function, verified via Fubini's theorem and standard Normal integration,

$$\Upsilon_n(\vartheta) := \frac{1}{\pi^n} \int_{\mathbb{R}_+^n} \mathbb{E} \left[\prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} \right] \prod_{i=1}^n dy_i = \mathbb{E} \left[\prod_{i=1}^n \frac{e^{-\beta^2 \tau_i}}{\sqrt{2\pi \tau_i}} \right] \leq \mathbb{E} \left[\prod_{i=1}^n \frac{1}{\sqrt{\tau_i}} \right]$$

These points lead to following representation of the likelihood.

Proposition 8.2 *Suppose that for fixed n , $\mathbb{E} \left[\prod_{i=1}^n \frac{1}{\sqrt{\tau_i}} \right] < \infty$, then the likelihood in Proposition 4.1 may be written as*

$$e^{\bar{A}\beta} \left[\Upsilon_n(\vartheta) - \frac{C_n(\mathbf{A}, \mathbf{p})}{\pi^n} \mathbb{E} [\Omega(Y_{1,n}, \dots, Y_{n,n}|\vartheta)] \right]$$

where

$$\Omega(y_1, \dots, y_n | \vartheta) = \frac{\mathbb{E} \left[\prod_{i=1}^n e^{-(y_i^2/2 + \beta^2/2)\tau_i} \right]}{\prod_{i=1}^n H(y_i | p_i)}$$

and the random vector $\{Y_{1,n}, \dots, Y_{n,n}\}$ has its joint distribution described by Proposition 8.1. \square

REMARK 21. Proposition 8.2 shows that one may approximate the likelihood by simulating random variables described in Proposition 8.1. Such an approach should work well with a Bayesian procedure. Methods to easily sample the random variables in Proposition 8.1, may be deduced from Devroye (1986a, b). In fact, through a personal communication with Luc Devroye we were informed that one at time sampling using the conditional distributions in Proposition 8.1 is routine as it constitutes essentially a sampling from a Normal density times a factor between 0 and 2. Hence rejection sampling is easy and furthermore the normalizing factor is not needed. One may also use other densities.

REMARK 22. Note that one needs also to evaluate $\Upsilon_n(\vartheta)$. Of course this can also be done by a Monte Carlo procedure using the density in Proposition 8.1.

9 Examples

In this section we will present some examples where we sketch out a few details related to our exposition. We will not concern ourselves too much with constants. Note that all the examples presented are *infinite-activity* processes. In the case where the distribution of $v(0)$ is not obvious we would simply approximate it when it is based on OU-FGGC models using Proposition 6.1, or choose an arbitrary law for $v(0)$ in a more general setting.

9.1 OU-Stable

Suppose that Z is stable subordinator of index α specified by $\psi(\omega) = \omega^\alpha$. Then it is known, or otherwise obvious, that $v(t)$ also has a stable law of index $0 < \alpha < 1$ with Lévy exponent $\omega^\alpha \int_0^1 u^{\alpha-1} du$. Notice that the Lévy exponent of the corresponding

$$\Phi(\omega) = \omega^\alpha \int_{e^{-a}}^1 (1-u)^\alpha u^{-1} du$$

corresponds also to a stable law of index α . Here, for simplicity of presentation, suppressing constants and setting $\beta = 0$ we may use Q_2 which is based on sampling the joint Laplace transform

$$(46) \quad e^{-[\sum_{i=1}^n y_i]^{\alpha}} \prod_{i=1}^n e^{-y_i^{2\alpha}}$$

Noting the simplicity of (46) it is good to recall that in general the densities of a stable law are only known in a complicated form. So here is a case where a Laplace transform approach is perhaps preferable despite the availability of the relevant densities. A nice exception to the preceding comment is when $\alpha = 1/2$ corresponding to an inverse Gamma law of index $\alpha = 1/2$. However in that case (46) is

$$e^{-[\sum_{i=1}^n y_i^2]^{1/2}} \prod_{i=1}^n e^{-y_i}.$$

For further simplification we may use the augmentation procedure described in section 7.2.2 applied to (46) to get

$$\prod_{i=1}^n e^{-y_i^{2\alpha}} e^{-vy_i^2} f_\alpha(v)$$

where f_α corresponds to a stable density. Note that although the stable density can be complicated there are many routines available to easily sample stable random variables.

REMARK 23. The Stable law process produces a log price process with heavy tails which may not be desirable for all applications. However see the work of Carr and Wu (2003). Additionally, we note that it would not be tremendously difficult to use Q_1 in this case.

9.2 IG-OU

This example is based on the calculations given in Barndorff-Nielsen and Shephard (2003, p. 292) where $v(t)$ has an Inverse Gaussian distribution. Here, letting C_1, C_2 denote constants and setting $\beta = 0$, by BNS(2003, eq. (54)) one has

$$\Phi(\omega) = -y_i^2 C_1 \int_0^{1-e^{-a}} (1-u)^{-1} u(1+C_2 y_i^2 u)^{-1/2} du.$$

BNS(2003) show that this can be written in terms of the hyperbolic arc-tangent function[see also Nicolato and Vernardos (2001) and Carr, Geman, Madan and Yor (2003)], we do not repeat that here. Note however by using the fact that $v(t)$ has a Inverse Gaussian distribution one can work with the augmented version of Q_2 which is proportional to

$$v^{-3/2} e^{-\frac{1}{2}[\gamma^2 v + \delta^2 v^{-1}]} \prod_{i=1}^n e^{-y_i^2 v} e^{-\Phi(\omega_i)}$$

for appropriate values of γ and δ and is not difficult to sample from.

9.3 OU-LogNormal

Suppose that Z is based on a LogNormal distribution with density

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2}(\log(x))^2} \text{ for } x > 0.$$

We have chosen this example because, despite the fact that it has a density with a nice closed form, its corresponding Lévy density ρ is unknown. Despite this we can still use a sampler based on Q_2 . This is because its Lévy exponent is given by

$$\psi(\omega) = -\log \left[\int_0^\infty e^{-\omega x} \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2}(\log(x))^2} dx \right].$$

This can be numerically approximated hence the relevant quantities $\Phi(\omega)$ can then be numerically approximated. Again this approximation should be done before the main Monte-Carlo procedure is used. Note that one would find it difficult or impossible to employ a series approximation in this case, as it depends on knowledge of ρ .

9.4 OU-FGGC where H is the Arcsine distribution

Here we close with one of the more interesting examples of known FGGC models. In this setting let H be the Arcsine law, that is there is a corresponding random variable W which is $\text{BETA}(1/2, 1/2)$. Cifarelli and Melilli (2000) show that in this setting for all $\theta > 0$, $M_{\theta H}$ is $\text{BETA}(\theta + 1/2, \theta + 1/2)$. Hence $Z(t) \stackrel{d}{=} T_{\theta t} B_{(\theta t + 1/2, \theta t + 1/2)}$, where here $B_{(\theta t + 1/2, \theta t + 1/2)}$ means a beta random variable with parameters indicated in the subscript. In this case the distribution of $R = (1 - U)W$ described in Proposition 6.1 has bounded support on $[0, 1]$. Hence we may apply the perfect sampler both for

option pricing and Monte Carlo methods specified according to Theorem 6.1 and Propositions 6.1 and 6.2. That is apply section 2.3.2 to sample from (Ξ_1, \mathbf{Y}) . To be clear given Y_i one may draw M_{ω_i} from $f_{M_{\omega_i}}$ by using Proposition 6.2 and creating $uM_{\theta a, -N} = 1$ and $lM_{\theta a, w_i, -N} = 0$, $B_{n, \theta a}$ is BETA $(1, \theta a)$ and $X_n \stackrel{d}{=} UW/(1 + W(1 - U)w_i)$. Then one draws W from the Arcsine law and U from F_a to get $X_{(n)}$. Draws from more complex densities for M_{w_i} can then be obtained by other standard methods. One can also work with the exact form of the densities via Cifarelli and Regazzini (1990).

References

- AÏT-SAHALIA, Y., MYKLAND, P. A. (2003). The effects of random and discrete sampling when estimating continuous-time diffusions. *Econometrica* **71** 483-549.
- AÏT-SAHALIA, Y., MYKLAND, P. A. (2004). Estimators of diffusions with randomly spaced discrete observations: a general theory. *Ann. Statist.* **32** 2186-2222.
- ANDREWS, G., ASKEY, R. AND ROY, R. (1999). *Special functions. Encyclopedia of Mathematics and its Applications, 71.* Cambridge University Press, Cambridge.
- BARNDORFF-NIELSEN, O.E. AND SHEPHARD, N. (2001a). Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. Royal Statist. Soc., Series B* **63** 167-241.
- BARNDORFF-NIELSEN, O.E. AND SHEPHARD, N. (2001b). Modelling by Lévy processes for financial econometrics. In Lévy processes. Theory and applications. Edited by Ole E. Barndorff-Nielsen, Thomas Mikosch and Sidney I. Resnick. p. 283-318. Birkhäuser Boston, Inc., Boston, MA.
- BARNDORFF-NIELSEN, O. E. AND SHEPHARD, N. (2003). Integrated OU processes and non-Gaussian OU-based stochastic volatility models. *Scand. J. Statist.* **30** 277-295.
- BENTH, F. E., KARLSEN, K. H. AND REIKVAM, K. (2003). Merton's portfolio optimization problem in a Black and Scholes market with non-Gaussian stochastic volatility of Ornstein-Uhlenbeck type. *Math. Finance* **13** 215-244.
- BLACK, F. AND SCHOLES, M. (1973). The pricing of options and corporate liabilities. *J. Polit. Econ.* **81** 637-654.
- BONDESSON, L. (1979). A general result on infinite divisibility. *Ann. Probab.* **7** 965-979.
- BONDESSON, L. (1992). Generalized gamma convolutions and related classes of distributions and densities. Lecture Notes in Statistics, 76. Springer-Verlag, New York.
- CARR, P., GEMAN, H., MADAN, D.B. AND YOR, M. (2003). Stochastic volatility for Lévy processes. *Math. Finance* **13** 345-382.
- CARR, P., GEMAN, H., MADAN, D.B. AND YOR, M. (2005). Self-Decomposability and Option Pricing. *Math. Finance* to appear.
- CARR, P. AND SCHRÖDER, M. (2004). Bessel processes, the integral of geometric Brownian motion, and Asian options. *Theor. Probab. Appl.* **48** 400-425.
- CARR, P. AND WU, L. (2003). The finite moment log stable process and option pricing. *Journal of Finance* **58** 753-778.
- CARR, P. AND WU, L. (2004). Time-changed Lévy processes and option pricing. *Journal of Financial Economics* **71** 113-141.
- CHEN, M-H., SHAO, Q-M., IBRAHIM, J.G. (2000). *Monte Carlo methods in Bayesian computation.* Springer Series in Statistics.. Springer-Verlag, New York.
- CIFARELLI, D. M. AND MELILLI, E. (2000). Some new results for Dirichlet priors. *Ann. Statist.* **28** 1390-1413.
- CIFARELLI, D. M. AND REGAZZINI, E. (1990). Distribution functions of means of a Dirichlet process. *Ann. Statist.* **18** 429-442.
- DEVROYE, L. (1986a). An automatic method for generating random variates with a given characteristic function. *SIAM J. Appl. Math.* **46** 698-719.
- DEVROYE, L. (1986b). *Nonuniform random variate generation..* Springer-Verlag, New York.

- DIACONIS, P. AND FREEDMAN, D. A. (1999). Iterated random functions. *Siam Rev.* **41** 45-76.
- DIACONIS, P. AND KEMPERMAN, J. (1996). Some new tools for Dirichlet priors. Bayesian Statistics 5 (J.M. Bernardo, J.O. Berger, A.P. Dawid and A.F.M. Smith eds.), Oxford University Press, pp. 97-106.
- DUAN, J. (1995). The GARCH option pricing model. *Math. Finance* **5** 13-32.
- DUFFIE, D., PAN, J. AND SINGLETON, K., (2000). Transform Analysis and Asset Pricing for Affine Jump Diffusions. *Econometrica* **68** 1343-1376.
- DYKSTRA, R. L. AND LAUD, P. W. (1981). A Bayesian nonparametric approach to reliability. *Ann. Statist.* **9** 356-367.
- EBERLEIN, E. (2001). Application of generalized hyperbolic Lévy motions to finance. In Lévy processes. Theory and applications. Edited by Ole E. Barndorff-Nielsen, Thomas Mikosch and Sidney I. Resnick. p. 319-336. Birkhäuser Boston, Inc., Boston, MA.
- ENGLE, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* **50** 987-1007.
- ERAKER, B., JOHANNES, M., AND POLSON, N. (2003). The impact of jumps in volatility and returns.. *Journal of Finance* **68** 1269-1300.
- FLAJOLET, P. AND SEDGEWICK, R. (2006). Analytic Combinatorics. Book to appear. Chapters available at <http://algo.inria.fr/flajolet/Publications/books.html>.
- GUGLIELMI, A., HOLMES, C.C., WALKER, S.G. . Perfect simulation involving functionals of a Dirichlet process. *J. Comput. Graph. Statist.* **11** 306-310.
- GRIFFIN, J. AND STEEL, M. (2005). Stochastic Volatility Inference with non-Gaussian Ornstein-Uhlenbeck Processes for Stochastic Volatility forthcoming Journal of Econometrics.
- GUGLIELMI, A., HOLMES, C.C., WALKER, S.G. . Perfect simulation involving functionals of a Dirichlet process. *J. Comput. Graph. Statist.* **11** 306-310.
- HJORT, N. L., AND ONGARO, A. (2005). Exact inference for random Dirichlet means.. *Stat. Inference Stoch. Process.* **8** 227-254.
- JAMES, L.F. (2005a). Functionals of Dirichlet processes, the Cifarelli-Regazzini identity and Beta-Gamma processes. *Ann. Statist.* **33** 647-660.
- JAMES, L.F. (2005b). Bayesian Poisson process partition calculus with an application to Bayesian Lévy moving averages.. *Ann. Statist.* **33** 1771-1799.
- JAMES, L.F. (2005c). Analysis of a class of likelihood based continuous time stochastic volatility models including Ornstein-Uhlenbeck models in financial economics. arXiv:math.ST/0503055.
- JEANBLANC, M., PITMAN, J. AND YOR, M. (2002). Self-similar processes with independent increments associated with Lévy and Bessel. *Stochastic Process. Appl.* **100** 223-231.
- JUREK, Z.J., VERVAAT, W. (1983 An integral representation for self-decomposable Banach space valued random variables).
- KONG, A., LIU, J.S., AND WONG, W. H. (1997). The properties of the cross-match estimate and split sampling. *Ann. Statist.* **25** 2410-2432.
- LIU, J.S. (2001). *Monte Carlo strategies in scientific computing*. Springer Series in Statistics. Springer-Verlag, New York.
- LUKACS, E.A. (1955). A characterization of the gamma distribution. *Ann. Math. Statist.* **26** 319-324.
- MADAN, D., CARR, P. AND CHANG, E. (1998). The variance gamma process and option pricing. *European Finance Rev.* **2** 79-105.
- MERTON, R. C. (1973). Theory of rational option pricing. *Bell J. Econ. Mgmt. Sci.* **4** 141-183.
- MAXIMON, L. C. (2003). The dilogarithm function for complex argument. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** 2807-2819.
- NICOLATO, E. AND VENARDOS, E. (2003). Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Math. Finance* **13** 445-466.

- PITMAN, J., (1999). Brownian motion, bridge, excursion, and meander characterized by sampling at independent uniform times. *Electron. J. Probab.* **4** 1-33.
- PROPP, J.G. AND WILSON, D. B. (1996). Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures Algorithms* **9** 223-252.
- ROBERTS, G. O., PAPASPILIOPOULOS, O. AND DELLAPORTAS, P. (2004). Bayesian inference for non-Gaussian Ornstein-Uhlenbeck stochastic volatility processes J. Royal Statist. Soc., Series B.
- SATO, K. (1999). *Lévy processes and infinitely divisible distributions. Translated from the 1990 Japanese original. Cambridge Studies in Advanced Mathematics, 68.* Cambridge University Press, Cambridge.
- THORIN, O. (1977). On the infinite divisibility of the lognormal distribution. *Scand. Actuar. J.* **3** 121-148.
- WATSON, G. N. (1966). *A treatise on the theory of Bessel functions. Paperback Edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge.*
- WOLFE, S. J. (1982). On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. *Stochastic Process. Appl.* **12** 301-312.
- VERSHIK, A.M., YOR, M. AND TSILEVICH, N.V. (2004). On the Markov-Krein identity and quasi-invariance of the gamma process. *J. Math. Sci.* **121** 2303-2310.
- YOR, M. (1992). On some exponential functionals of Brownian motion. *Adv. in Appl. Probab.* **24** 509-531.

LANCELOT F. JAMES
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF INFORMATION AND SYSTEMS MANAGEMENT
CLEAR WATER BAY, KOWLOON
HONG KONG
lancelot@ust.hk